PUNJAB BOARD NOTES

MATHEMATICS (EM)

9TH CLASS

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UNIT 1

MATRICES AND DETERMINANTS

Unit Outlines

- *Introduction to Matrices*
- *Types of Matrices*
- *Addition and Subtraction of Matrices*
- *Multiplication of Matrices*
- *Multiplicative Inverse of a Matrix*
- *Solution of Simultaneous Linear Equations*

STUDENTS LEARNING OUTCOMES

After studying this unit the students will be able to:

- Define
  - a matrix with real entries and relate its rectangular layout (formation) with real life
  - rows and columns of a matrix
  - the order of a matrix
  - equality of two matrices
- Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, identity matrix, scalar matrix, diagonal matrix, transpose of a matrix, symmetric and skew-symmetric matrices.
- Know whether the given matrices are suitable for addition/subtraction
- Add and subtract matrices
- Multiply a matrix by a real number
- Verify commutative and associative laws under addition
- Define additive identity of a matrix
- Find additive inverse of a matrix
- Know whether the given matrices are suitable for multiplication
- Multiply two (or three) matrices
- Verify associative law under multiplication
- Verify distributive laws
- Show with the help of an example that commutative law under multiplication does not hold in general (i.e., $AB \neq BA$)
- Define multiplicative identity of a matrix.
- Verify the result $(AB)^t = B^t A^t$
- Define the determinant of a square matrix
- Evaluate determinant of a matrix
- Define singular and non-singular matrices
- Define adjoint of a matrix
- Find multiplicative inverse of a non-singular matrix A and verify that

$AA^{-1} = I = A^{-1}A$ where I is the identity matrix
- Use adjoint method to calculate inverse of a non-singular matrix
- Verify the result $(AB)^{-1} = B^{-1}A^{-1}$
- Solve a system of two linear equations and related real life problems in two unknowns using
  - Matrix inversion method
  - Cramer's rule

Matrix:

A rectangular array or a formation of a collection of real numbers, say 0, 1, 2, 3, 4 and 7, such as $\begin{matrix} 1 & 3 & 4 \\ 7 & 2 & 0 \end{matrix}$ and then enclosed by brackets [ ].

For example, $\begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$, etc.

Remember:

The matrices are denoted conventionally by the capital letters A, B, C, ..., M, N, etc. of the English alphabets.

Rows and Columns of a Matrix:

In matrix, the entries presented in horizontal way are called "rows".

In matrix, the entries presented in vertical way are called "columns".

For example, $\underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}}_{\text{Columns}} \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \Big\} \text{Rows}$

Order of a Matrix:

The number of rows and columns in a matrix specifies its order. If a matrix M has m rows and n columns then M is said to be of order, m – by – n.

For example, $M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ is of order 2 – by – 3.

$N = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 2 & 3 & 7 \end{bmatrix}$ is of order 3 – by – 3.

$P = \begin{bmatrix} 3 & 2 & 5 \end{bmatrix}$ is of order 1 – by – 3.

Equal Matrices:

Let A and B be two matrices, then A is said to be equal to B, and denoted by A = B, if and only if

(i) the order of A = the order of B. **(ii)** their corresponding elements are equal.

Examples:

(1) $A = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2+1 \\ -4 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$ are equal matrices.

ختم نبوت ﷺ زندہ باد **عظمت صحابہ زندہ باد**

السلام علیکم ورحمۃ اللہ وبرکاتہ:

معزز ممبران: آپ کا وٹس ایپ گروپ ایڈ من **"اردو بکس"** آپ سے مخاطب ہے۔

آپ تمام ممبران سے گزارش ہے کہ:

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❖ گروپ میں معزز، پڑھے لکھے، سلجھے ہوئے ممبرز موجود ہیں اخلاقیات کی پابندی کریں اور گروپ رولز کو فالو کریں بصورت دیگر معزز ممبرز کی بہتری کی خاطر ریمو و کر دیا جائے گا۔

❖ کوئی بھی ممبر کسی بھی ممبر کو انباکس میں میسیج، مس کال، کال نہیں کرے گا۔ رپورٹ پر فوری ریمو و کر کے کاروائی عمل میں لائے جائے گی۔

❖ ہمارے کسی بھی گروپ میں سیاسی و فرقہ واریت کی بحث کی قطعاً کوئی گنجائش نہیں ہے۔

❖ اگر کسی کو بھی گروپ کے متعلق کسی قسم کی شکایت یا تجویز کی صورت میں ایڈ من سے رابطہ کیجئے۔

❖ سب سے اہم بات:

گروپ میں کسی بھی قادیانی، مرزائی، احمدی، گستاخِ رسول، گستاخِ امہات المؤمنین، گستاخِ صحابہ و خلفائے راشدین حضرت ابو بکر صدیق، حضرت عمر فاروق، حضرت عثمان غنی، حضرت علی المرتضیٰ، حضرت حسنین کریمین رضوان اللہ تعالیٰ اجمعین، گستاخ اہلبیت یا ایسے غیر مسلم جو اسلام اور پاکستان کے خلاف پراپیگنڈا میں مصروف ہیں یا ان کے روحانی و ذہنی سپورٹرز کے لئے کوئی گنجائش نہیں ہے لہذا ایسے اشخاص بالکل بھی گروپ جوائن کرنے کی زحمت نہ کریں۔ معلوم ہونے پر فوراً ریمو و کر دیا جائے گا۔

❖ تمام کتب انٹرنیٹ سے تلاش / ڈاؤنلوڈ کر کے **فری آف کاسٹ** وٹس ایپ گروپ میں شیئر کی جاتی ہیں۔ جو کتاب نہیں ملتی اس کے لئے معذرت کر لی جاتی ہے۔ جس میں محنت بھی صَرف ہوتی ہے لیکن ہمیں آپ سے صرف دعاؤں کی درخواست ہے۔

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❖ **لیڈیز کے لئے الگ گروپ کی سہولت موجود ہے جس کے لئے ویریفکیشن ضروری ہے۔**

❖ اردو کتب / عمران سیریز یا سٹڈی گروپ میں ایڈ ہونے کے لئے ایڈ من سے وٹس ایپ پر بذریعہ میسیج رابطہ کریں اور جواب کا انتظار فرمائیں۔ برائے مہربانی اخلاقیات کا خیال رکھتے ہوئے موبائل پر کال یا ایم ایس کرنے کی کوشش ہر گز نہ کریں۔ ورنہ گروپس سے تو ریمو و کیا ہی جائے گا بلاک بھی کیا جائے گا۔

نوٹ: ہمارے کسی گروپ کی کوئی فیس نہیں ہے۔ سب فی سبیل اللہ ہے

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اللہ تبارک تعالیٰ ہم سب کا حامی و ناصر ہو

We see that:

(i) the order of matrix A = the order of matrix B.

(ii) their corresponding elements are equal.

Hence, A = B.

(2) $L = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $M = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ are not equal matrices.

We see that order of matrix L = the order of matrix M but entries in the second row and second column are not equal.

Hence, $L \neq M$.

(3) $P = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \end{bmatrix}$ are not equal matrices.

We see that order of $P \neq$ the order of Q.

Hence, $P \neq Q$.

Solved Exercise 1.1

1. Find the order of the following matrices.

$$A = \begin{bmatrix} 2 & 3 \\ -5 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 4 \end{bmatrix}, D = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}, E = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}, F = [2],$$

$$G = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}, H = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \end{bmatrix}$$

Solution:

$A = \begin{bmatrix} 2 & 3 \\ -5 & 6 \end{bmatrix}$

In this matrix

Number of rows = 2

Number of columns = 2

Therefore order of matrix A is 2–by–2

$B = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}$

In this matrix

Number of rows = 2

Number of columns = 2

Therefore order of matrix B is 2–by–2

$C = [2 \quad 4]$

$$D = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$$

$$E = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

$$F = [2]$$

$$G = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \end{bmatrix}$$

2. Which of the following matrices are equal?

$$A=[3]\ ,\ B=[3\quad 5]\ ,\ C=[5-2],\ D=[5\quad 3], E=\begin{bmatrix}4&0\\6&2\end{bmatrix}, F=\begin{bmatrix}2\\6\end{bmatrix},$$

$$G=\begin{bmatrix}3-1\\3+3\end{bmatrix}, H=\begin{bmatrix}4&0\\6&2\end{bmatrix}, I=[3\quad 3+2], J=\begin{bmatrix}2+2&2-2\\2+4&2+0\end{bmatrix}$$

Solution:

$$A=[3]_{(1,1)}, B=[3\quad 5]_{(1,2)}, C=[5-2]=[3]_{(1,1)}, D=[5\ 3]_{(1,2)}, E=\begin{bmatrix}4&0\\6&2\end{bmatrix}_{(2,2)}, F=\begin{bmatrix}2\\6\end{bmatrix}_{(2,1)}$$

$$G=\begin{bmatrix}3-1\\3+3\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}_{(2,1)}, H=\begin{bmatrix}4&0\\6&2\end{bmatrix}_{(2,2)}, I=[3\ \ 3+2]=[3\quad 5]_{(1,2)}$$

$$J=\begin{bmatrix}2+2&2-2\\2+4&2+0\end{bmatrix}=\begin{bmatrix}4&0\\6&2\end{bmatrix}_{(2,2)}$$

For Equal Matrices:

Two matrices are said to be equal if and only if their orders and corresponding entries are equal. So

$A = C$, $B = I$, $E = H = J$, $F = G$.

3. Find the values of a, b, c and d which satisfy the matrix equation.

$$\begin{bmatrix}a+c&a+2b\\c-1&4d-6\end{bmatrix}=\begin{bmatrix}0&-7\\3&2d\end{bmatrix}$$

Solution:

$$\Rightarrow \quad a+c=0 \quad \text{———} \quad (1)$$
$$a+2b=-7 \quad \text{———} \quad (2)$$
$$c-1=3 \quad \text{———} \quad (3)$$
$$4d-6=2d \quad \text{———} \quad (4)$$

From eq. (3), we have

$$c-1=3$$
$$c=3+1$$
$$c=4$$

From eq. (4), we have

$$4d-6=2d$$
$$4d-2d=6$$
$$2d=6$$
$$d=3$$

Put $c = 4$ in eq. (1), we have

$$a+4=0$$
$$a=-4$$

Put $a = -4$ in eq. (2), we have

$$
\begin{aligned}
-4+2b &= -7 \\
2b &= -7+4 \\
2b &= -3 \\
b &= -1.5
\end{aligned}
$$

So, $a=-4$, $b=-1.5$, $c=4$, $d=3$.

Types of Matrices

(i) Row Matrix:

A matrix is called a row matrix if it has only one row.

For example, $M=[2 \quad -7]$, $N=[1 \quad -1]$, etc.

(ii) Column Matrix:

A matrix is called a column matrix if it has only on column.

For example, $M=\begin{bmatrix}1\\0\end{bmatrix}, N=\begin{bmatrix}2\\0\\1\end{bmatrix}$, etc.

(iii) Rectangular Matrix:

A matrix M is called rectangular, if the number of rows of M is not equal to the number of columns of M.

For example, $A=\begin{bmatrix}1&2\\1&1\\2&3\end{bmatrix}, B=\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}$, etc.

(iv) Square Matrix:

A matrix is called a square matrix, if its number of rows is equal to its number of columns.

For example, $A=\begin{bmatrix}2&-1\\0&3\end{bmatrix}, B=\begin{bmatrix}1&2&3\\-1&0&-2\\0&1&3\end{bmatrix}, c=[3]$, etc.

(v) Null or Zero Matrix:

A matrix is called a null or zero matrix, if each of its entries is 0.

For example, $\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$, etc.

(vi) Transpose of a Matrix:

A matrix obtained by changing the rows into columns or columns into rows of a matrix is called transpose of that matrix. If A is a matrix, then its transpose is denoted by A^t:

For example:

(i) If $A=\begin{bmatrix}1&2&3\\2&1&0\\-1&4&-4\end{bmatrix}$, then $A^t=\begin{bmatrix}1&2&-1\\2&1&4\\3&0&-4\end{bmatrix}$

(ii) If $B=\begin{bmatrix}1 & 0 & 2\\ 2 & -1 & 3\end{bmatrix}$, then $B^t=\begin{bmatrix}1 & 2\\ 0 & -1\\ 2 & 3\end{bmatrix}$

(iii) If $C=\begin{bmatrix}0 & 1\end{bmatrix}$, then $C^t=\begin{bmatrix}0\\ 1\end{bmatrix}$

(vii) Negative of a Matrix:

Let A be a matrix. Then its negative, $-A$ is obtained by changing the signs of all the entries of A.

For example, If $A=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}$, than $-A=\begin{bmatrix}-1 & 2\\ -3 & -4\end{bmatrix}$

(viii) Symmetric Matrix:

A square matrix is symmetric, if it is equal to its transpose. i.e., matrix A is symmetric if $A^t = A$.

For example, If $M=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 0\end{bmatrix}$, then $M^t=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 0\end{bmatrix}$

As $M^t = M$. So M is a symmetric matrix.

(ix) Skew- Symmetric Matrix:

A square matrix A is said to be skew – symmetric, if $A^t = -A$.

For example, $A=\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 1\\ -3 & -1 & 0\end{bmatrix}$, then $A^t=\begin{bmatrix}0 & -2 & -3\\ 2 & 0 & -1\\ 3 & 1 & 0\end{bmatrix}=-\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 1\\ -3 & -1 & 0\end{bmatrix}=-A$

Since, $A^t = -A$. So A is a skew- symmetric Matrix.

(x) Diagonal Matrix:

A square matrix A is called a diagonal matrix, if at least any one of the entries of its diagonal is not zero and non-diagonal entries are zero.

For example, $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$, etc

(xi) Scalar Matrix:

A diagonal matrix is called a scalar matrix, if all the diagonal entries are same and non-zero.

For example, $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$, etc.

(xii) Identity Matrix:

A diagonal matrix is called identity (unit) matrix, if all diagonal entries are '1' and

it is denoted by I.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

Remember:

(i) The scalar matrix and identity matrix are diagonal matrices.

(ii) Every diagonal matrix is not a scalar or identity matrix.

Solved Exercise 1.2

1. From the following matrices, identify unit matrices, row matrices, column matrices and null matrices.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 \end{bmatrix}, F = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Solution:

Matrix A is a null matrix because each of its entries is 0.
Matrix B is a row matrix because it has only one row.
Matrix C is a column matrix because it has only one column.
Matrix D is a unit matrix because all diagonal entries of the matrix D are 1.
Matrix E is a null matrix because each of its entries is 0.
Matrix F is a column matrix because it has only one column.

2. From the following matrices, identify

(a) Square matrices **(b)** Rectangular matrices
(c) Row matrices **(d)** Column matrices
(e) Identity matrices **(f)** Null matrices

(i) $\begin{bmatrix} -8 & 2 & 7 \\ 12 & 0 & 4 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **(v)** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & 10 & -1 \end{bmatrix}$ **(vii)** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **(viii)** $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **(ix)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution:

(a)	Square matrices	(iii), (iv), (viii)
(b)	Rectangular matrices	(i), (ii), (v), (vi), (vii), (ix)
(c)	Row matrices	(vi)
(d)	Column matrices	(ii), (vii)
(e)	Identity matrices	(iv)
(f)	Null matrices	(ix)

3. From the following matrices, identify diagonal, scalar and unit (identity) matrices.

$$A=\begin{bmatrix}4&0\\0&4\end{bmatrix},\quad B=\begin{bmatrix}2&0\\0&-1\end{bmatrix},C=\begin{bmatrix}1&0\\0&1\end{bmatrix},D=\begin{bmatrix}3&0\\0&0\end{bmatrix},E=\begin{bmatrix}5-3&0\\0&1+1\end{bmatrix}$$

Solution: Matrix A is a scalar matrix because all the diagonal entries of the matrix A are same and non-zero.

Matrix B is a diagonal matrix because atleast any one of the entries of its diagonal is not zero and non-diagonal entries are zero.

Matrix C is a unit matrix because all diagonal entries are 1.

Matrix D is a diagonal matrix because atleast any one of the entries of its diagonal is not zero and non-diagonal entries are zero.

Matrix E is a scalar matrix because all the diagonal entries are same and non-zero.

4. Find negative of matrices A, B, C, D and E when:

$$A=\begin{bmatrix}1\\0\\-1\end{bmatrix},B=\begin{bmatrix}3&-1\\2&1\end{bmatrix},C=\begin{bmatrix}2&6\\3&2\end{bmatrix},D=\begin{bmatrix}-3&2\\-4&5\end{bmatrix},E=\begin{bmatrix}1&-5\\2&3\end{bmatrix}$$

Solution: Negative of matrices A, B, C, D and E when:

$$-A=\begin{bmatrix}-1\\0\\1\end{bmatrix},-B=\begin{bmatrix}-3&1\\-2&-1\end{bmatrix},-C=\begin{bmatrix}-2&-6\\-3&-2\end{bmatrix},-D=\begin{bmatrix}3&-2\\4&-5\end{bmatrix},-E=\begin{bmatrix}-1&5\\-2&-3\end{bmatrix}$$

5. Find the transpose of each of the following matrices:

$$A=\begin{bmatrix}0\\1\\-2\end{bmatrix},\quad B=\begin{bmatrix}5&1&-6\end{bmatrix},C=\begin{bmatrix}1&2\\2&-1\\3&0\end{bmatrix},D=\begin{bmatrix}2&3\\0&5\end{bmatrix},\quad E=\begin{bmatrix}2&3\\-4&5\end{bmatrix},\quad F=\begin{bmatrix}1&2\\3&4\end{bmatrix}$$

Solution: $A=\begin{bmatrix}0\\1\\-2\end{bmatrix}\Rightarrow A^t=\begin{bmatrix}0&1&-2\end{bmatrix}$, $\quad B=\begin{bmatrix}5&1&-6\end{bmatrix}\Rightarrow B^t=\begin{bmatrix}5\\1\\-6\end{bmatrix}$

$$C=\begin{bmatrix}1&2\\2&-1\\3&0\end{bmatrix}\Rightarrow C^t=\begin{bmatrix}1&2&3\\2&-1&0\end{bmatrix},\qquad D=\begin{bmatrix}2&3\\0&5\end{bmatrix}\Rightarrow D^t=\begin{bmatrix}2&0\\3&5\end{bmatrix}$$

$$E=\begin{bmatrix}2&3\\-4&5\end{bmatrix}\Rightarrow E^t=\begin{bmatrix}2&-4\\3&5\end{bmatrix},\qquad F=\begin{bmatrix}1&2\\3&4\end{bmatrix}\Rightarrow F^t=\begin{bmatrix}1&3\\2&4\end{bmatrix}$$

6. Verify that if $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $B=\begin{bmatrix}1&1\\2&0\end{bmatrix}$, then

(i) $(A^t)^t=A$ **(ii)** $(B^t)^t=B$

Solution: (i) $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}\Rightarrow A^t=\begin{bmatrix}1&0\\2&1\end{bmatrix}\Rightarrow (A^t)^t=\begin{bmatrix}1&2\\0&1\end{bmatrix}=A$

Hence $(A^t)^t=A$

(ii) $B=\begin{bmatrix}1&1\\2&0\end{bmatrix}\Rightarrow B^t=\begin{bmatrix}1&2\\1&0\end{bmatrix}\Rightarrow (B^t)^t=\begin{bmatrix}1&1\\2&0\end{bmatrix}=B$

Hence $(B^t)^t=B$

ADDITION AND SUBTRACTION OF MATRICES

Addition of Matrices:

Let A and B be any two matrices with real number entries. The matrices A and B are conformable for addition, if they have same order.

For example, if $A=\begin{bmatrix}2&3&0\\1&0&6\end{bmatrix}$ and $B=\begin{bmatrix}-2&3&4\\1&2&3\end{bmatrix}$ are conformable for addition.

Addition of A and B, written as A + B, is obtained by adding the entries of the matrix A to the corresponding entries of the matrix B.

e.g., $A+B=\begin{bmatrix}2&3&0\\1&0&6\end{bmatrix}+\begin{bmatrix}-2&3&4\\1&2&3\end{bmatrix}=\begin{bmatrix}2+(-2)&3+3&0+4\\1+1&0+2&6+3\end{bmatrix}=\begin{bmatrix}0&6&4\\2&2&9\end{bmatrix}$

Subtraction of Matrices:

If A and B are two matrices of same order, then subtraction of matrix B from matrix A is obtained by subtracting the entries of matrix B from the corresponding entries of the matrix A and it is denoted by A − B.

For example, if $A=\begin{bmatrix}2&3&4\\1&5&0\end{bmatrix}$ and $B=\begin{bmatrix}0&2&2\\-1&4&3\end{bmatrix}$ are conformable for subtraction.

Then $A-B=\begin{bmatrix}2&3&4\\1&5&0\end{bmatrix}-\begin{bmatrix}0&2&2\\-1&4&3\end{bmatrix}=\begin{bmatrix}2-0&3-2&4-2\\1-(-1)&5-4&0-3\end{bmatrix}=\begin{bmatrix}2&1&2\\2&1&-3\end{bmatrix}$

Some solved examples regarding addition and subtraction are given below.

(a) If $A=\begin{bmatrix}1&2&7\\0&-1&3\\2&5&1\end{bmatrix}$ and $B=\begin{bmatrix}0&3&4\\1&-1&2\\5&-2&7\end{bmatrix}$

Then $A+B=\begin{bmatrix}1&2&7\\0&-1&3\\2&5&1\end{bmatrix}+\begin{bmatrix}0&3&4\\1&-1&2\\5&-2&7\end{bmatrix}=\begin{bmatrix}1+0&2+3&7+4\\0+1&-1+(-1)&3+2\\2+5&5+(-2)&1+7\end{bmatrix}=\begin{bmatrix}1&5&11\\1&-2&5\\7&3&8\end{bmatrix}$

and $A+(-B)=\begin{bmatrix}1&2&7\\0&-1&3\\2&5&1\end{bmatrix}+\begin{bmatrix}0&-3&-4\\-1&1&-2\\-5&2&-7\end{bmatrix}=\begin{bmatrix}1-0&2-3&7-4\\0-1&-1-(-1)&3-2\\2-5&5-(-2)&1-7\end{bmatrix}=\begin{bmatrix}1&-1&3\\-1&0&1\\-3&7&-6\end{bmatrix}$

(b) If $A=\begin{bmatrix}1&2\\-1&3\\0&2\end{bmatrix}$ and $B=\begin{bmatrix}2&3\\1&-2\\3&4\end{bmatrix}$

Then $A+B=\begin{bmatrix}1&2\\-1&3\\0&2\end{bmatrix}+\begin{bmatrix}2&3\\1&-2\\3&4\end{bmatrix}=\begin{bmatrix}1+2&2+3\\-1+1&3+(-2)\\0+3&2+4\end{bmatrix}=\begin{bmatrix}3&5\\0&1\\3&6\end{bmatrix}$

Then $A-B=\begin{bmatrix}1&2\\-1&3\\0&2\end{bmatrix}-\begin{bmatrix}2&3\\1&-2\\3&4\end{bmatrix}=\begin{bmatrix}1-2&2-3\\-1-1&3-(-2)\\0-3&2-4\end{bmatrix}=\begin{bmatrix}-1&-1\\-2&5\\-3&-2\end{bmatrix}$

Multiplication of a Matrix by a Real Number:

Let A be any matrix and the real number K be a scalar. Then the scalar multiplication of matrix A with K is obtained by multiplying each entry of matrix A with K. It is denoted by KA.

For example, If $A=\begin{bmatrix}1 & -1 & 4\\ 2 & -1 & 0\\ -1 & 3 & 2\end{bmatrix}$ be a matrix of order 3-by-3 and k=-2 be a real number.

Then $kA=(-2)A=(-2)\begin{bmatrix}1 & -1 & 4\\ 2 & -1 & 0\\ -1 & 3 & 2\end{bmatrix}=\begin{bmatrix}(-2)(1) & (-2)(-1) & (-2)(4)\\ (-2)(2) & (-2)(-1) & (-2)(0)\\ (-2)(-1) & (-2)(3) & (-2)(2)\end{bmatrix}=\begin{bmatrix}-2 & 2 & -8\\ -4 & 2 & 0\\ 2 & -6 & -4\end{bmatrix}$

Commutative and Associative laws of Addition of Matrices:

Commutative law under Addition:

If A and B are two matrices of same order, then A + B = B + A is called commutative law under addition.

For example, If $A=\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}$

Then $A+B=\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}+\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}=\begin{bmatrix}2+3 & 3-2 & 0+5\\ 5-1 & 6+4 & 1+1\\ 2+4 & 1+2 & 3-4\end{bmatrix}$

$=\begin{bmatrix}5 & 1 & 5\\ 4 & 10 & 2\\ 6 & 3 & -1\end{bmatrix}$ ____(1)

And $B+A=\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}+\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}=\begin{bmatrix}3+2 & -2+3 & 5+0\\ -1+5 & 4+6 & 1+1\\ 4+2 & 2+1 & -4+3\end{bmatrix}$

$=\begin{bmatrix}5 & 1 & 5\\ 4 & 10 & 2\\ 6 & 3 & -1\end{bmatrix}$ ____(2)

From (1) and (2), we have

$A+B=B+A$

Hence proved.

Associative law under addition:

If A, B and C are three matrices of same order, such that

(A + B) + C = A + (B+C) is called associative law under addition.

For example, If $A=\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 2 & 3\\ -2 & 0 & 4\\ 1 & 2 & 0\end{bmatrix}$

$$\text{Then } (A+B)+C = \left(\begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}+\begin{bmatrix}3&-2&5\\-1&4&1\\4&2&-4\end{bmatrix}\right)+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}2+3&3-2&0+5\\5-1&6+4&1+1\\2+4&1+2&3-4\end{bmatrix}+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}=\begin{bmatrix}5&1&5\\4&10&2\\6&3&-1\end{bmatrix}+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}5+1&1+2&5+3\\4-2&10+0&2+4\\6+1&3+2&-1+0\end{bmatrix}=\begin{bmatrix}6&3&8\\2&10&6\\7&5&-1\end{bmatrix} \text{ ......(1)}$$

$$A+(B+C) = \begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}+\left(\begin{bmatrix}3&-2&5\\-1&4&1\\4&2&-4\end{bmatrix}+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}+\begin{bmatrix}3+1&-2+2&5+3\\-1-2&4+0&1+4\\4+1&2+2&-4+0\end{bmatrix}=\begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}+\begin{bmatrix}4&0&8\\-3&4&5\\5&4&-4\end{bmatrix}$$

$$=\begin{bmatrix}2+4&3+0&0+8\\5-3&6+4&1+5\\2+5&1+4&3-4\end{bmatrix}=\begin{bmatrix}6&3&8\\2&10&6\\7&5&-1\end{bmatrix} \text{ ......(2)}$$

From (1) and (2), we have

$$(A+B)+C = A+(B+C)$$

Hence proved.

Additive Identity of a Matrix:

If A and B are two matrices of same order and $A + B = A = B + A$

Then matrix B is called additive identity of matrix A.

For any matrix A and zero matrix O of same order, O is called additive identity of A as: $A + O = A = O + A$

For example, If $A = \begin{bmatrix}1&2\\3&5\end{bmatrix}$ and $O = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

$$\text{Then } A+O = \begin{bmatrix}1&2\\3&5\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}1+0&2+0\\3+0&5+0\end{bmatrix}=\begin{bmatrix}1&2\\3&5\end{bmatrix}=A$$

$$\text{And } O+A = \begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}1&2\\3&5\end{bmatrix}=\begin{bmatrix}0+1&0+2\\0+3&0+5\end{bmatrix}=\begin{bmatrix}1&2\\3&5\end{bmatrix}=A$$

Additive inverse of a Matrix:

If A and B are two matrices of same order. Such that $A + B = O = B + A$

Then A and B are called additive inverse of each other.

Additive inverse of any matrix A is obtained by changing to negative of the symbols (entries) of each non-zero entry of A.

For example, If $A = \begin{bmatrix}1&2&1\\0&1&-2\\3&1&0\end{bmatrix}$

Then $B=(-A)=-\begin{bmatrix}1&2&1\\0&-1&-2\\3&1&0\end{bmatrix}=\begin{bmatrix}-1&-2&-1\\0&1&2\\-3&-1&0\end{bmatrix}$ is the additive inverse of A.

It can be verified as:

$$A+B=\begin{bmatrix}1&2&1\\0&-1&-2\\3&1&0\end{bmatrix}+\begin{bmatrix}-1&-2&-1\\0&1&2\\-3&-1&0\end{bmatrix}=\begin{bmatrix}1+(-1)&2+(-2)&1+(-1)\\0+0&-1+1&-2+2\\3+(-3)&1+(-1)&0+0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$$

$$B+A=\begin{bmatrix}-1&-2&-1\\0&1&2\\-3&-1&0\end{bmatrix}+\begin{bmatrix}1&2&1\\0&-1&-2\\3&1&0\end{bmatrix}=\begin{bmatrix}-1+1&-2+2&-1+1\\0+0&1+(-1)&2+(-2)\\-3+3&-1+1&0+0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$$

Since $A + B = O = B + A$

Therefore, A and B is additive inverse of each other.

Solved Exercise 1.3

1. Which of the following matrices are conformable for addition?

$$A=\begin{bmatrix}2&1\\-1&3\end{bmatrix}, B=\begin{bmatrix}3\\1\end{bmatrix}, C=\begin{bmatrix}1&0\\2&-1\\1&-2\end{bmatrix}, D=\begin{bmatrix}2+1\\3\end{bmatrix}, E=\begin{bmatrix}-1&0\\1&2\end{bmatrix}, F=\begin{bmatrix}3&2\\1+1&-4\\3+2&2+1\end{bmatrix}$$

Solution:

Matrices A and E are conformable for addition because they have same order.
Matrices B and D are conformable for addition because they have same order.
Matrices C and F are conformable for addition because they have same order.

2. Find the additive inverse of following matrices.

$$A=\begin{bmatrix}2&4\\-2&1\end{bmatrix}, B=\begin{bmatrix}1&0&-1\\2&-1&3\\3&-2&1\end{bmatrix}, C=\begin{bmatrix}4\\-2\end{bmatrix}, D=\begin{bmatrix}1&0\\-3&-2\\2&1\end{bmatrix}, E=\begin{bmatrix}1&0\\0&1\end{bmatrix}, F=\begin{bmatrix}\sqrt{3}&1\\-1&\sqrt{2}\end{bmatrix}$$

Solution:

$A=\begin{bmatrix}2&4\\-2&1\end{bmatrix}$, then $-A=\begin{bmatrix}-2&-4\\2&-1\end{bmatrix}$ is the additive inverse of A.

$B=\begin{bmatrix}1&0&-1\\2&-1&3\\3&-2&1\end{bmatrix}$, then $-B=\begin{bmatrix}-1&0&1\\-2&1&-3\\-3&2&-1\end{bmatrix}$ is the additive inverse of B.

$C=\begin{bmatrix}4\\-2\end{bmatrix}$, then $-C=\begin{bmatrix}-4\\2\end{bmatrix}$ is the additive inverse of C.

$D=\begin{bmatrix}1&0\\-3&-2\\2&1\end{bmatrix}$, then $-D=\begin{bmatrix}-1&0\\3&2\\-2&-1\end{bmatrix}$ is the additive inverse of D.

$E=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, then $-E=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ is the additive inverse of E.

$F=\begin{bmatrix}\sqrt{3}&1\\-1&\sqrt{2}\end{bmatrix}$, then $-F=\begin{bmatrix}-\sqrt{3}&-1\\1&-\sqrt{2}\end{bmatrix}$ is the additive inverse of F.

3. If $A=\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}, B=\begin{bmatrix}1\\ -1\end{bmatrix}, C=\begin{bmatrix}1 & -1 & 2\end{bmatrix}, D=\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{bmatrix}$, then find,

(i) $A+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

Solution: $A+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1+1 & 2+1\\ 2+1 & 1+1\end{bmatrix}=\begin{bmatrix}0 & 3\\ 3 & 2\end{bmatrix}$

(ii) $B+\begin{bmatrix}-2\\ 3\end{bmatrix}$

Solution: $B+\begin{bmatrix}-2\\ 3\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}+\begin{bmatrix}-2\\ 3\end{bmatrix}=\begin{bmatrix}1+(-2)\\ -1+3\end{bmatrix}=\begin{bmatrix}-1\\ 2\end{bmatrix}$

(iii) $C+\begin{bmatrix}-2 & 1 & 3\end{bmatrix}$

Solution: $C+\begin{bmatrix}-2 & 1 & 3\end{bmatrix}=\begin{bmatrix}1 & -1 & 2\end{bmatrix}+\begin{bmatrix}-2 & 1 & 3\end{bmatrix}=\begin{bmatrix}1+(-2) & -1+1 & 2+3\end{bmatrix}=\begin{bmatrix}-1 & 0 & 5\end{bmatrix}$

(iv) $D+\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}$

Solution: $D+\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{bmatrix}+\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}=\begin{bmatrix}1+0 & 2+1 & 3+0\\ -1+2 & 0+0 & 2+1\end{bmatrix}=\begin{bmatrix}1 & 3 & 3\\ 1 & 0 & 3\end{bmatrix}$

(v) $2A$

Solution: $2A=2\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}=\begin{bmatrix}(-1)(2) & (2)(2)\\ (2)(2) & (1)(2)\end{bmatrix}=\begin{bmatrix}-2 & 4\\ 4 & 2\end{bmatrix}$

(vi) $(-1)B$

Solution: $(-1)B=(-1)\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}(1)(-1)\\ (-1)(-1)\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}$

(vii) $(-2)C$

Solution: $(-2)C=(-2)\begin{bmatrix}1 & -1 & 2\end{bmatrix}=\begin{bmatrix}(-2)(1) & (-2)(-1) & (-2)(2)\end{bmatrix}=\begin{bmatrix}-2 & 2 & -4\end{bmatrix}$

(viii) $3D$

Solution: $3D=3\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{bmatrix}=\begin{bmatrix}(3)(1) & (3)(2) & (3)(3)\\ (3)(-1) & (3)(0) & (3)(2)\end{bmatrix}=\begin{bmatrix}3 & 6 & 9\\ -3 & 0 & 6\end{bmatrix}$

(ix) $3C$

Solution: $3C=3\begin{bmatrix}1 & -1 & 2\end{bmatrix}=\begin{bmatrix}(3)(1) & (3)(-1) & (3)(2)\end{bmatrix}=\begin{bmatrix}3 & -3 & 6\end{bmatrix}$

4. Perform the indicated operations and simplify the following.

(i) $\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 2\\ 3 & 0\end{bmatrix}\right)+\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$

Solution: $\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 2\\ 3 & 0\end{bmatrix}\right)+\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1+0 & 0+2\\ 0+3 & 1+0\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$

$=\begin{bmatrix}1+1 & 2+1\\ 3+1 & 1+0\end{bmatrix}=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}$

(ii) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\left(\begin{bmatrix}0 & 2\\ 3 & 0\end{bmatrix}-\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right)$

Solution: $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\left(\begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}-\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0-1 & 2-1\\3-1 & 0-0\end{bmatrix}$

$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}-1 & 1\\2 & 0\end{bmatrix}=\begin{bmatrix}1-1 & 0+1\\0+2 & 1+0\end{bmatrix}=\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}$

(iii) $\begin{bmatrix}2 & 3 & 1\end{bmatrix}+\left(\begin{bmatrix}1 & 0 & 2\end{bmatrix}-\begin{bmatrix}2 & 2 & 2\end{bmatrix}\right)$

Solution: $\begin{bmatrix}2 & 3 & 1\end{bmatrix}+\left(\begin{bmatrix}1 & 0 & 2\end{bmatrix}-\begin{bmatrix}2 & 2 & 2\end{bmatrix}\right)=\begin{bmatrix}2 & 3 & 1\end{bmatrix}+\begin{bmatrix}1-2 & 0-2 & 2-2\end{bmatrix}$

$=\begin{bmatrix}2 & 3 & 1\end{bmatrix}+\begin{bmatrix}-1 & -2 & 0\end{bmatrix}$

$=\begin{bmatrix}2+(-1) & 3+(-2) & 1+0\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\end{bmatrix}$

(iv) $\begin{bmatrix}1 & 2 & 3\\-1 & -1 & -1\\0 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 1 & 1\\2 & 2 & 2\\3 & 3 & 3\end{bmatrix}$

Solution: $\begin{bmatrix}1 & 2 & 3\\-1 & -1 & -1\\0 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 1 & 1\\2 & 2 & 2\\3 & 3 & 3\end{bmatrix}=\begin{bmatrix}1+1 & 2+1 & 3+1\\-1+2 & -1+2 & -1+2\\0+3 & 1+3 & 2+3\end{bmatrix}=\begin{bmatrix}2 & 3 & 4\\1 & 1 & 1\\3 & 4 & 5\end{bmatrix}$

(v) $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0 & -2\\-2 & -1 & 0\\0 & 2 & -1\end{bmatrix}$

Solution: $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0 & -2\\-2 & -1 & 0\\0 & 2 & -1\end{bmatrix}=\begin{bmatrix}1+1 & 2+0 & 3+(-2)\\2+(-2) & 3+(-1) & 1+0\\3+0 & 1+2 & 2+(-1)\end{bmatrix}=\begin{bmatrix}2 & 2 & 1\\0 & 2 & 1\\3 & 3 & 1\end{bmatrix}$

(vi) $\left(\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}+\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\right)+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

Solution: $\left(\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}+\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\right)+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}=\begin{bmatrix}1+2 & 2+1\\0+1 & 1+0\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}=\begin{bmatrix}3 & 3\\1 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$=\begin{bmatrix}3+1 & 3+1\\1+1 & 1+1\end{bmatrix}=\begin{bmatrix}4 & 4\\2 & 2\end{bmatrix}$

5. For the matrices $A=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}, B=\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$ **and** $C=\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}$ **verify the following rules.**

(i) $A+C=C+A$

Solution: L. H. S. $= A + C$

$=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}=\begin{bmatrix}1+(-1) & 2+0 & 3+0\\2+0 & 3+(-2) & 1+3\\1+1 & -1+1 & 0+2\end{bmatrix}=\begin{bmatrix}0 & 2 & 3\\2 & 1 & 4\\2 & 0 & 2\end{bmatrix}$ ____(1)

R.H.S. $= C + A =\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}=\begin{bmatrix}-1+1 & 0+2 & 0+3\\0+2 & -2+3 & 3+1\\1+1 & 1+(-1) & 2+0\end{bmatrix}=\begin{bmatrix}0 & 2 & 3\\2 & 1 & 4\\2 & 0 & 2\end{bmatrix}$ ____(2)

From (1) and (2), we have $A+C=C+A$

Hence proved.

(ii) $\mathbf{A + B = B + A}$

Solution: L. H. S. = A + B

$$=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}=\begin{bmatrix}1+1&2+(-1)&3+1\\2+2&3+(-2)&1+2\\1+3&-1+1&0+3\end{bmatrix}=\begin{bmatrix}2&1&4\\4&1&3\\4&0&3\end{bmatrix}\text{\_\_\_(1)}$$

R.H.S. = B + A

$$=\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}+\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}=\begin{bmatrix}1+1&-1+2&1+3\\2+2&-2+3&2+1\\3+1&1+(-1)&3+0\end{bmatrix}=\begin{bmatrix}2&1&4\\4&1&3\\4&0&3\end{bmatrix}\text{\_\_\_(2)}$$

From (1) and (2) we have: A + B = B + A

Hence proved.

(iii) $\mathbf{B + C = C + B}$

Solution: L. H. S. = B + C

$$=\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}=\begin{bmatrix}1+(-1)&-1+0&1+0\\2+0&-2+(-2)&2+3\\3+1&1+1&3+2\end{bmatrix}=\begin{bmatrix}0&-1&1\\2&-4&5\\4&2&5\end{bmatrix}\text{\_\_\_(1)}$$

R.H.S. = C + B

$$=\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}=\begin{bmatrix}-1+1&0+(-1)&0+1\\0+2&-2+(-2)&3+2\\1+3&1+1&2+3\end{bmatrix}=\begin{bmatrix}0&-1&1\\2&-4&5\\4&2&5\end{bmatrix}\text{\_\_\_(2)}$$

From (1) and (2), we have: B + C = C + B

Hence proved.

(iv) $\mathbf{A + (B + A) = 2A + B}$

Solution: L. H. S. = A + (B + A)

$$=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\left(\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}+\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}\right)=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}1+1&-1+2&1+3\\2+2&-2+3&2+1\\3+1&1+(-1)&3+0\end{bmatrix}$$

$$=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}2&1&4\\4&1&3\\4&0&3\end{bmatrix}=\begin{bmatrix}1+2&2+1&3+4\\2+4&3+1&1+3\\1+4&-1+0&0+3\end{bmatrix}=\begin{bmatrix}3&3&7\\6&4&4\\5&-1&3\end{bmatrix}\text{\_\_\_(1)}$$

R.H.S. = 2 A + B

$$=2\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}=\begin{bmatrix}2&4&6\\4&6&2\\2&-2&0\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}$$

$$=\begin{bmatrix}2+1&4+(-1)&6+1\\4+2&6+(-2)&2+2\\2+3&-2+1&0+3\end{bmatrix}=\begin{bmatrix}3&3&7\\6&4&4\\5&-1&3\end{bmatrix}\text{\_\_\_(2)}$$

From (1) and (2) we have: A + (B + A) = 2 A + B

Hence proved.

(V) $(C - B) + A = C + (A - B)$

Solution: L. H. S. $= (C - B) + A$

$$= \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}\right) + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1-1 & 0-(-1) & 0-1 \\ 0-2 & -2-(-2) & 3-2 \\ 1-3 & 1-1 & 2-3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & -1 \\ -2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2+1 & 1+2 & -1+3 \\ -2+2 & 0+3 & 1+1 \\ -2+1 & 0+(-1) & -1+0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 2 \\ 0 & 3 & 2 \\ -1 & -1 & -1 \end{bmatrix} \text{\_\_\_\_(1)}$$

R.H.S. $= C + (A - B)$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix} + \left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1-1 & 2-(-1) & 3-1 \\ 2-2 & 3-(-2) & 1-2 \\ 1-3 & -1-1 & 0-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 2 \\ 0 & 5 & -1 \\ -2 & -2 & -3 \end{bmatrix} = \begin{bmatrix} -1+0 & 0+3 & 0+2 \\ 0+0 & -2+5 & 3+(-1) \\ 1+(-2) & 1+(-2) & 2+(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 2 \\ 0 & 3 & 2 \\ -1 & -1 & -1 \end{bmatrix} \text{\_\_\_\_(2)}$$

From (1) and (2) we have: $(C - B) + A = C + (A - B)$

Hence proved.

(vi) $2A + B = A + (A + B)$

Solution: L. H. S. $= 2A + B$

$$= 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 2 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 4+(-1) & 6+1 \\ 4+2 & 6+(-2) & 2+2 \\ 2+3 & -2+1 & 0+3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 7 \\ 6 & 4 & 4 \\ 5 & -1 & 3 \end{bmatrix} \text{\_\_\_\_(1)}$$

R.H.S. $= A + (A+B)$

$$=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\left(\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}\right)$$

$$=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}1+1&2+(-1)&3+1\\2+2&3+(-2)&1+2\\1+3&-1+1&0+3\end{bmatrix}$$

$$=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}2&1&4\\4&1&3\\4&0&3\end{bmatrix}=\begin{bmatrix}1+2&2+1&3+4\\2+4&3+1&1+3\\1+4&-1+0&0+3\end{bmatrix}=\begin{bmatrix}3&3&7\\6&4&4\\5&-1&3\end{bmatrix}\text{—(2)}$$

From (1) and (2) we have: $2A+B=A+A+B$

Hence proved.

(vii) $(C-B)-A=(C-A)-B$

Solution: L. H. S. = $(C-B)-A$

$$=\left(\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}-\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}\right)-\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}=\begin{bmatrix}-1-1&0-(-1)&0-1\\0-2&-2-(-2)&3-2\\1-3&1-1&2-3\end{bmatrix}-\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}$$

$$=\begin{bmatrix}-2&1&-1\\-2&0&1\\-2&0&-1\end{bmatrix}-\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}=\begin{bmatrix}-2-1&1-2&-1-3\\-2-2&0-3&1-1\\-2-1&0-(-1)&-1-0\end{bmatrix}=\begin{bmatrix}-3&-1&-4\\-4&-3&0\\-3&1&-1\end{bmatrix}\text{—(1)}$$

R.H.S. = $(C-A)-B$

$$=\left(\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}-\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}\right)-\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}=\begin{bmatrix}-1-1&0-2&0-3\\0-2&-2-3&3-1\\1-1&1-(-1)&2-0\end{bmatrix}-\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}$$

$$=\begin{bmatrix}-2&-2&-3\\-2&-5&2\\0&2&2\end{bmatrix}-\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}=\begin{bmatrix}-2-1&-2-(-1)&-3-1\\-2-2&-5-(-2)&2-2\\0-3&2-1&2-3\end{bmatrix}=\begin{bmatrix}-3&-1&-4\\-4&-3&0\\-3&1&-1\end{bmatrix}\text{—(2)}$$

From (1) and (2) we have: $(C-B)-A=(C-A)-B$

Hence proved.

(viii) $(A+B)+C=A+(B+C)$

Solution: L. H. S. = $(A+B)+C$

$$=\left(\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}\right)+\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}$$

$$=\begin{bmatrix}1+1&2+(-1)&3+1\\2+2&3+(-2)&1+2\\1+3&-1+1&0+3\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}=\begin{bmatrix}2&1&4\\4&1&3\\4&0&3\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}$$

$$=\begin{bmatrix}2+(-1)&1+0&4+0\\4+0&1+(-2)&3+3\\4+1&0+1&3+2\end{bmatrix}=\begin{bmatrix}1&1&4\\4&-1&6\\5&1&5\end{bmatrix}\text{—(1)}$$

R. H. S. = $A+(B+C)$

$$=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\left(\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}+\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\begin{bmatrix}1+(-1) & (-1)+0 & 1+0\\2+0 & (-2)+(-2) & 2+3\\3+1 & 1+1 & 3+2\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\begin{bmatrix}0 & -1 & 1\\2 & -4 & 5\\4 & 2 & 5\end{bmatrix}$$

$$=\begin{bmatrix}1+0 & 2+(-1) & 3+1\\2+2 & 3+(-4) & 1+5\\1+4 & -1+2 & 0+5\end{bmatrix}=\begin{bmatrix}1 & 1 & 4\\4 & -1 & 6\\5 & 1 & 5\end{bmatrix}\_\_(2)$$

From (1) and (2) we have: $(A+B)+C=A+(B+C)$
Hence proved.

(ix) $A+(B-C)=(A-C)+B$

Solution: L. H. S. $=A+(B-C)$

$$=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\left(\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}-\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\begin{bmatrix}1-(-1) & -1-0 & 1-0\\2-0 & -2-(-2) & 2-3\\3-1 & 1-1 & 3-2\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\begin{bmatrix}2 & -1 & 1\\2 & 0 & -1\\2 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1+2 & 2+(-1) & 3+1\\2+2 & 3+0 & 1+(-1)\\1+2 & -1+0 & 0+1\end{bmatrix}=\begin{bmatrix}3 & 1 & 4\\4 & 3 & 0\\3 & -1 & 1\end{bmatrix}\_\_(1)$$

R. H. S. $=(A-C)+B$

$$=\left(\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}-\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}\right)+\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1-(-1) & 2-0 & 3-0\\2-0 & 3-(-2) & 1-3\\1-1 & -1-1 & 0-2\end{bmatrix}+\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}=\begin{bmatrix}2 & 2 & 3\\2 & 5 & -2\\0 & -2 & -2\end{bmatrix}+\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2+1 & 2+(-1) & 3+1\\2+2 & 5+(-2) & -2+2\\0+3 & -2+1 & -2+3\end{bmatrix}=\begin{bmatrix}3 & 1 & 4\\4 & 3 & 0\\3 & -1 & 1\end{bmatrix}\_\_(2)$$

From (1) and (2) we have: $A+(B-C)=(A-C)+B$
Hence proved.

(x) $2A+2B=2(A+B)$

Solution: L. H. S. $=2A+2B$

$$=2\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+2\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}=\begin{bmatrix}2 & 4 & 6\\4 & 6 & 2\\2 & -2 & 0\end{bmatrix}+\begin{bmatrix}2 & -2 & 2\\4 & -4 & 4\\6 & 2 & 6\end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 4+(-2) & 6+2 \\ 4+4 & 6+(-4) & 2+4 \\ 2+6 & -2+2 & 0+6 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 8 \\ 8 & 2 & 6 \\ 8 & 0 & 6 \end{bmatrix} \_\_(1)$$

R. H. S. $= 2(A + B)$

$$= 2\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}\right)$$

$$= 2\left(\begin{bmatrix} 1+1 & 2+(-1) & 3+1 \\ 2+2 & 3+(-2) & 1+2 \\ 1+3 & -1+1 & 0+3 \end{bmatrix}\right) = 2\begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 3 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 8 \\ 8 & 2 & 6 \\ 8 & 0 & 6 \end{bmatrix} \_\_(2)$$

From (1) and (2) we have: $2A + 2B = 2(A + B)$
Hence proved.

6. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 7 \\ -3 & 8 \end{bmatrix}$, find (i) $3A - 2B$ (ii) $2A^t - 3B^t$.

Solution: (i) $3A - 2B$

$$= 3\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 0 & 7 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 14 \\ -6 & 16 \end{bmatrix} = \begin{bmatrix} 3-0 & -6-14 \\ 9-(-6) & 12-16 \end{bmatrix} = \begin{bmatrix} 3 & -20 \\ 15 & -4 \end{bmatrix}$$

(ii) $2A^t - 3B^t$

Now $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 7 \\ -3 & 8 \end{bmatrix}$, then $B^t = \begin{bmatrix} 0 & -3 \\ 7 & 8 \end{bmatrix}$

So, $2A^t - 3B^t = 2\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} - 3\begin{bmatrix} 0 & -3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 0 & -9 \\ 21 & 24 \end{bmatrix}$

$$= \begin{bmatrix} 2-0 & 6-(-9) \\ -4-21 & 8-24 \end{bmatrix} = \begin{bmatrix} 2 & 15 \\ -25 & -16 \end{bmatrix}$$

7. If $2\begin{bmatrix} 2 & 4 \\ -3 & a \end{bmatrix} + 3\begin{bmatrix} 1 & b \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$, then find a and b.

Solution: $2\begin{bmatrix} 2 & 4 \\ -3 & a \end{bmatrix} + 3\begin{bmatrix} 1 & b \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 8 \\ -6 & 2a \end{bmatrix} + \begin{bmatrix} 3 & 3b \\ 24 & -12 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4+3 & 8+3b \\ -6+24 & 2a-12 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8+3b \\ 18 & 2a-12 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$$

$\Rightarrow \quad 8 + 3b = 10 \quad$ and $\quad 2a - 12 = 1$

$$3b = 10 - 8 \qquad 2a = 1 + 12$$
$$3b = 2 \qquad 2a = 13$$
$$b = \frac{2}{3} \qquad a = \frac{13}{2}$$

8. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, then verify that

(i) $(A + B)^t = A^t + B^t$

Solution: L.H.S $= (A + B)^t$

Now $A + B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+1 \\ 0+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

So, $(A + B)^t = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ —— (1)

R. H. S. $= A^t + B^t$

Now $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow B^t = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

So, $A^t + B^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+1 & 0+2 \\ 2+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ —— (2)

From (1) and (2), we have : $(A + B)^t = A^t + B^t$

Hence proved.

(ii) $(A - B)^t = A^t - B^t$

Solution: L. H. S. $= (A - B)^t$

Now $A - B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 2-1 \\ 0-2 & 1-0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$

So, $(A - B)^t = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$ —— (1)

R. H. S. $= A^t - B^t$

Now $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow B^t = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

So, $A^t - B^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 0-2 \\ 2-1 & 1-0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$ —— (2)

From (1) and (2), we have : $(A - B)^t = A^t - B^t$

Hence proved.

(iii) $A + A^t$ is symmetric

Solution: We have to prove that

$(A + A^t)^t = A + A^t$

Now $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

So, $A+A^t=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}=\begin{bmatrix}1+1 & 2+0\\0+2 & 1+1\end{bmatrix}=\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$

Now, $(A+A^t)^t=\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$

$(A+A^t)^t=A+A^t$

Hence proved.

(iv) $A-A^t$ is skew symmetric

Solution: We have to prove that

$(A-A^t)^t=-(A-A^t)$

Now $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\Rightarrow A^t=\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$

So, $A-A^t=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}-\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}=\begin{bmatrix}1-1 & 2-0\\0-2 & 1-1\end{bmatrix}=\begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}$

$(A-A^t)^t=\begin{bmatrix}0 & -2\\2 & 0\end{bmatrix}$

Now, $(A-A^t)^t=\begin{bmatrix}0 & -2\\2 & 0\end{bmatrix}=-\begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}$

$(A-A^t)^t=-(A-A^t)$

Hence proved.

(v) $B+B^t$ is symmetric

Solution: We have to prove that

$(B+B^t)^t=B+B^t$

Now $B=\begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}\Rightarrow B^t=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}$

So, $B+B^t=\begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}+\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}=\begin{bmatrix}1+1 & 1+2\\2+1 & 0+0\end{bmatrix}=\begin{bmatrix}2 & 3\\3 & 0\end{bmatrix}$

Now, $(B+B^t)^t=\begin{bmatrix}2 & 3\\3 & 0\end{bmatrix}$

$(B+B^t)^t=B+B^t$

Hence proved.

(vi) $B-B^t$ is skew symmetric

Solution: We have to prove that

$(B-B^t)^t=-(B-B^t)$

Now, $B=\begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}\Rightarrow B^t=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}$

So, $B - B^t = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-2 \\ 2-1 & 0-0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Now, $(B - B^t)^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$(B - B^t)^t = -(B - B^t)$

Hence proved.

MULTIPLICATION OF MATRICES

Two matrices A and B are conformable for multiplication, giving product AB if the number of columns of A is equal to the number of rows of B.

For Example, let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Here the number of columns of A is equal to the number of rows of B. So A and B matrices are conformable for multiplication.

Multiplication of two matrices is explained by following examples.

Example: (i) If $A = [1 \quad 2]$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

Then $AB = [1 \quad 2]\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = [1\times2+2\times3 \quad 1\times0+2\times1] = [2+6 \quad 0+2] = [8 \quad 2]$

Example: (ii) If $A = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1\times(-1)+3\times3 & 1\times0+3\times2 \\ 2\times(-1)+(-3)\times3 & 2\times0+(-3)\times2 \end{bmatrix} = \begin{bmatrix} -1+9 & 0+6 \\ -2-9 & 0-6 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ -11 & -6 \end{bmatrix}$

Associative law under Multiplication:

If A, B and C are three matrices conformable for multiplication then associative law under multiplication is given as.

$(AB)C = A(BC)$

For example, if $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$, then

L.H.S = (AB)C

$= \left(\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\right)\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2\times0+3\times3 & 2\times1+3\times1 \\ -1\times0+0\times3 & -1\times1+0\times1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0+9 & 2+3 \\ 0+0 & -1+0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 9\times2+5\times(-1) & 9\times2+5\times0 \\ 0\times2+(-1)\times(-1) & 0\times2+(-1)\times0 \end{bmatrix} = \begin{bmatrix} 18-5 & 18+0 \\ 0+1 & 0-0 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 1 & 0 \end{bmatrix}$ —— (1)

R.H.S. = A (BC)

$$=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\left(\begin{bmatrix}0&1\\3&1\end{bmatrix}\begin{bmatrix}2&2\\-1&0\end{bmatrix}\right)=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}0\times2+1\times(-1)&0\times2+1\times0\\3\times2+1\times(-1)&3\times2+1\times0\end{bmatrix}$$

$$=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}0-1&0+0\\6-1&6+0\end{bmatrix}=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}-1&0\\5&6\end{bmatrix}$$

$$=\begin{bmatrix}2\times(-1)+3\times5&2\times0+3\times6\\(-1)\times(-1)+0\times5&(-1)\times0+0\times6\end{bmatrix}=\begin{bmatrix}-2+15&0+18\\1+0&0+0\end{bmatrix}=\begin{bmatrix}13&18\\1&0\end{bmatrix}\text{---}(2)$$

From (1) and (2), we have: (AB) C = A (BC)
Hence proved.

Distributive law of Multiplication over Addition and Subtraction:

(a) **Let A, B and C be three matrices then distributive law of multiplication over addition are given below.**

(i) A (B + C) = AB + AC (Left Distributive Law)
(ii) (A + B) C = AC + BC (Right Distributive Law)

For example, let $A=\begin{bmatrix}2&3\\-1&0\end{bmatrix}$, $B=\begin{bmatrix}0&1\\3&1\end{bmatrix}$ and $C=\begin{bmatrix}2&2\\-1&0\end{bmatrix}$, then prove distributive law.

A (B + C) = AB + AC.

L.H.S.= A (B + C)

$$=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\left(\begin{bmatrix}0&1\\3&1\end{bmatrix}+\begin{bmatrix}2&2\\-1&0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}0+2&1+2\\3+(-1)&1+0\end{bmatrix}=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}2&3\\2&1\end{bmatrix}$$

$$=\begin{bmatrix}2\times2+3\times2&2\times3+3\times1\\(-1)\times2+0\times2&(-1)\times3+0\times1\end{bmatrix}=\begin{bmatrix}4+6&6+3\\-2+0&3+0\end{bmatrix}=\begin{bmatrix}10&9\\-2&-3\end{bmatrix}\text{---}(1)$$

R.H.S. = AB + AC

$$=\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\3&1\end{bmatrix}+\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}2&2\\-1&0\end{bmatrix}$$

$$=\begin{bmatrix}2\times0+3\times3&2\times1+3\times1\\(-1)\times0+0\times3&(-1)\times1+0\times1\end{bmatrix}+\begin{bmatrix}2\times2+3\times(-1)&2\times2+3\times0\\(-1)\times2+0\times(-1)&(-1)\times2+0\times0\end{bmatrix}$$

$$=\begin{bmatrix}0+9&2+3\\0+0&-1+0\end{bmatrix}+\begin{bmatrix}4-3&4+0\\-2-0&-2+0\end{bmatrix}$$

$$=\begin{bmatrix}9&5\\0&-1\end{bmatrix}+\begin{bmatrix}1&4\\-2&-2\end{bmatrix}=\begin{bmatrix}9+1&5+4\\0+(-2)&-1+(-2)\end{bmatrix}=\begin{bmatrix}10&9\\-2&-3\end{bmatrix}\text{---}(2)$$

From (1) and (2), we have A (B + C) = AB + AC
Hence proved.

(b) **Similarly the distributive law of multiplication over subtraction are as follow.**

(i) A (B – C) = AB – AC (ii) (A – B) C = AC + BC

For example, let $A=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}, B=\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}$ and $C=\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$, then prove distributive law.

$A(B-C)=AB-AC.$

L.H.S. $=A(B-C)$

$$=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\left(\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}-\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right)=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}-1-2 & 1-1\\1-1 & 0-2\end{bmatrix}=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}-3 & 0\\0 & -2\end{bmatrix}$$

$$=\begin{bmatrix}2\times(-3)+3\times0 & 2\times0+3\times(-2)\\0\times(-3)+1\times0 & 0\times0+1\times(-2)\end{bmatrix}=\begin{bmatrix}-6+0 & 0-6\\0+0 & 0-2\end{bmatrix}=\begin{bmatrix}-6 & -6\\0 & -2\end{bmatrix}\text{—(1)}$$

R.H.S. $=AB-AC$

$$=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}-\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}2\times(-1)+3\times1 & 2\times1+3\times0\\0\times(-1)+1\times1 & 0\times1+1\times0\end{bmatrix}-\begin{bmatrix}2\times2+3\times1 & 2\times1+3\times2\\0\times2+1\times1 & 0\times1+1\times2\end{bmatrix}$$

$$=\begin{bmatrix}-2+3 & 2+0\\0+1 & 0+0\end{bmatrix}-\begin{bmatrix}4+3 & 2+6\\0+1 & 0+2\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}-\begin{bmatrix}7 & 8\\1 & 2\end{bmatrix}=\begin{bmatrix}1-7 & 2-8\\1-1 & 0-2\end{bmatrix}=\begin{bmatrix}-6 & -6\\0 & -2\end{bmatrix}\text{—(2)}$$

From (1) ad (2), we have: $A(B-C)=AB-AC$

Hence proved.

Commutative law of Multiplication of Matrices.

Consider the matrices

$A=\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$, then

$$AB=\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}=\begin{bmatrix}0\times1+1\times0 & 0\times0+1\times(-2)\\2\times1+3\times0 & 2\times0+3\times(-2)\end{bmatrix}$$

$$=\begin{bmatrix}0+0 & 0-2\\2+0 & 0-6\end{bmatrix}=\begin{bmatrix}0 & -2\\2 & -6\end{bmatrix}\text{—(1)}$$

And $$BA=\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}=\begin{bmatrix}1\times0+0\times2 & 1\times1+0\times3\\0\times0+(-2)\times2 & 0\times1+(-2)\times3\end{bmatrix}$$

$$=\begin{bmatrix}0+0 & 1+0\\0-4 & 0-6\end{bmatrix}=\begin{bmatrix}0 & 1\\-4 & -6\end{bmatrix}\text{—(2)}$$

From (1) ad (2), we have: $AB\neq BA$

Hence proved.

Commutative law under multiplication in matrices does not hold in general.

i.e., if A and B are two matrices then $AB\neq BA$.

Commutative law under multiplication holds in particular case.

For example, If $A=\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 0\\0 & 4\end{bmatrix}$, then

$$AB=\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-3 & 0\\0 & 4\end{bmatrix}=\begin{bmatrix}2\times(-3)+0\times0 & 2\times0+0\times4\\0\times(-3)+1\times0 & 0\times0+1\times4\end{bmatrix}$$

$$=\begin{bmatrix}-6+0 & 0+0\\0+0 & 0+4\end{bmatrix}=\begin{bmatrix}-6 & 0\\0 & 4\end{bmatrix}\text{---}(1)$$

$$BA=\begin{bmatrix}-3 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-3\times2+0\times0 & -3\times0+0\times1\\0\times2+4\times0 & 0\times0+4\times1\end{bmatrix}$$

$$=\begin{bmatrix}-6+0 & 0+0\\0+0 & 0+4\end{bmatrix}=\begin{bmatrix}-6 & 0\\0 & 4\end{bmatrix}\text{---}(2)$$

From (1) ad (2), we have: $AB=BA$

Hence proved.

Multiplicative Identity of a Matrix:

Let A be a matrix. Another matrix B is called the identity matrix of A under multiplication, if

$AB=A=BA$

For example, if $A=\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then

$$AB=\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}1\times1+2\times0 & 1\times0+2\times1\\0\times1+(-3)\times0 & 0\times0+(-3)\times1\end{bmatrix}$$

$$=\begin{bmatrix}1+0 & 0+2\\0+0 & 0-3\end{bmatrix}=\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}=A\text{---}(1)$$

$$BA=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+0+0 & 1\times2+0(-3)\\0\times1+1\times0 & 0\times2+1\times(-3)\end{bmatrix}=\begin{bmatrix}1+0 & 2+0\\0+0 & 0-3\end{bmatrix}=\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}=A\text{---}(2)$$

From (1) and (2), we have: $AB=A=BA$

Hence proved.

Verification of $(AB)^t=B^tA^t$

If A, B are two matrices and A^t, B^t are their respective transpose, then

$(AB)^t=B^t\,A^t$

For example, if $A=\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\-2 & 0\end{bmatrix}$

L.H.S.= $(AB)^t$

Now $AB=\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\-2 & 0\end{bmatrix}=\begin{bmatrix}2\times1+1\times(-2) & 2\times3+1\times0\\0\times1+(-1)\times(-2) & 0\times3+(-1)\times0\end{bmatrix}$

$$=\begin{bmatrix}2-2 & 6+0\\0+2 & 0+0\end{bmatrix}=\begin{bmatrix}0 & 6\\2 & 0\end{bmatrix}$$

$$(AB)^t=\begin{bmatrix}0 & 2\\6 & 0\end{bmatrix}\text{——}(1)$$

R.H.S. = B^tA^t

Now $A=\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}\Rightarrow A^t=\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\-2 & 0\end{bmatrix}\Rightarrow B^t=\begin{bmatrix}1 & -2\\3 & 0\end{bmatrix}$

$$B^tA^t=\begin{bmatrix}1 & -2\\3 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}=\begin{bmatrix}1\times2+(-2)\times1 & 1\times0+(-2)\times(-1)\\3\times2+0\times1 & 3\times0+0\times(-1)\end{bmatrix}$$

$$=\begin{bmatrix}2-2 & 0+2\\6+0 & 0+0\end{bmatrix}=\begin{bmatrix}0 & 2\\6 & 0\end{bmatrix}\text{——}(2)$$

From (1) and (2), we have $(AB)^t=B^t\,A^t$

Solved Exercise 1.4

1. Which of the following product of matrices is conformable for multiplication?

(i) $\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$ (iii) $\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}$

(iv) $\begin{bmatrix}1 & 2\\0 & -1\\-1 & -2\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\end{bmatrix}$ (v) $\begin{bmatrix}3 & 2 & 1\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 2\\-2 & 3\end{bmatrix}$

Solution:

(i) $\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}$

Conformable for multiplication because the number of columns of first matrix is equal to the number of rows of second matrix.

(ii) $\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$

Conformable for multiplication because the number of columns of first matrix is equal to the number of **rows** of second matrix.

(iii) $\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}$

Not - Conformable for multiplication because the number of columns of first matrix is not equal to the number of rows of second matrix.

(iv) $\begin{bmatrix}1 & 2\\0 & -1\\-1 & -2\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\end{bmatrix}$

Conformable for multiplication because the number of columns of first matrix is equal to the **number** of rows of second matrix.

(v) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & 3 \end{bmatrix}$

Conformable for multiplication because the number of columns of first matrix is equal to the number of rows of second matrix.

2. If $A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 6 \\ 5 \end{bmatrix}$, find (i) AB (ii) BA (if possible)

Solution: (i) $AB=\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}_{(2,2)}\begin{bmatrix} 6 \\ 5 \end{bmatrix}_{(2,1)}=\begin{bmatrix} 3\times 6+0\times 5 \\ -1\times 6+2\times 5 \end{bmatrix}=\begin{bmatrix} 18+0 \\ -6+10 \end{bmatrix}=\begin{bmatrix} 18 \\ 4 \end{bmatrix}$

(ii) $BA=\begin{bmatrix} 6 \\ 5 \end{bmatrix}_{(2,1)}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}_{(2,2)}$

Not - Conformable for multiplication because the number of columns of first matrix is not equal to the number of rows of second matrix.

3. Find the following products.

(i) $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}=\begin{bmatrix} 1\times 4+2\times 0 \end{bmatrix}=\begin{bmatrix} 4+0 \end{bmatrix}=\begin{bmatrix} 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix}=\begin{bmatrix} 1\times 5+2\times(-4) \end{bmatrix}=\begin{bmatrix} 5-8 \end{bmatrix}=\begin{bmatrix} -3 \end{bmatrix}$

(iii) $\begin{bmatrix} -3 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Solution: $\begin{bmatrix} -3 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}=\begin{bmatrix} -3\times 4+0\times 0 \end{bmatrix}=\begin{bmatrix} -12+0 \end{bmatrix}=\begin{bmatrix} -12 \end{bmatrix}$

(iv) $\begin{bmatrix} 6 & -0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Solution: $\begin{bmatrix} 6 & -0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}=\begin{bmatrix} 6\times 4+0\times 0 \end{bmatrix}=\begin{bmatrix} 24+0 \end{bmatrix}=\begin{bmatrix} 24 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 0 & -4 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 0 & -4 \end{bmatrix}=\begin{bmatrix} 1\times 4+2\times 0 & 1\times 5+2\times(-4) \\ (-3)\times 4+0\times 0 & (-3)\times 5+0\times(-4) \\ 6\times 4+(-1)\times 0 & 6\times 5+(-1)\times(-4) \end{bmatrix}$

$$=\begin{bmatrix}4+0 & 5-8\\ -12+0 & -15+0\\ 24+0 & 30+4\end{bmatrix}=\begin{bmatrix}4 & -3\\ -12 & -15\\ 24 & 34\end{bmatrix}$$

4. Multiply the following matrices.

(a) $\begin{bmatrix}2 & 3\\ 1 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 0\end{bmatrix}$

Solution: $\begin{bmatrix}2 & 3\\ 1 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 0\end{bmatrix}=\begin{bmatrix}2\times2+3\times3 & 2\times(-1)+3\times0\\ 1\times2+1\times3 & 1\times(-1)+1\times0\\ 0\times2+(-2)\times3 & 0\times(-1)+(-2)\times0\end{bmatrix}$

$$=\begin{bmatrix}4+9 & -2+0\\ 2+3 & -1+0\\ 0-6 & 0+0\end{bmatrix}=\begin{bmatrix}13 & -2\\ 5 & -1\\ -6 & 0\end{bmatrix}$$

(b) $\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\\ -1 & 1\end{bmatrix}$

Solution: $\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1\times1+2\times3+3\times(-1) & 1\times2+2\times4+3\times1\\ 4\times1+5\times3+6\times(-1) & 4\times2+5\times4+6\times1\end{bmatrix}$

$$=\begin{bmatrix}1+6-3 & 2+8+3\\ 4+15-6 & 8+20+6\end{bmatrix}=\begin{bmatrix}4 & 13\\ 13 & 34\end{bmatrix}$$

(c) $\begin{bmatrix}1 & 2\\ 3 & 4\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$

Solution: $\begin{bmatrix}1 & 2\\ 3 & 4\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}1\times1+2\times4 & 1\times2+2\times5 & 1\times3+2\times6\\ 3\times1+4\times4 & 3\times2+4\times5 & 3\times3+4\times6\\ (-1)\times1+1\times4 & (-1)\times2+1\times5 & (-1)\times3+1\times6\end{bmatrix}$

$$=\begin{bmatrix}1+8 & 2+10 & 3+12\\ 3+16 & 6+20 & 9+24\\ -1+4 & -2+5 & -3+6\end{bmatrix}=\begin{bmatrix}9 & 12 & 15\\ 19 & 26 & 33\\ 3 & 3 & 3\end{bmatrix}$$

(d) $\begin{bmatrix}8 & 5\\ 6 & 4\end{bmatrix}\begin{bmatrix}2 & -5/2\\ -4 & 4\end{bmatrix}$

Solution: $\begin{bmatrix}8 & 5\\ 6 & 4\end{bmatrix}\begin{bmatrix}2 & -\frac{5}{2}\\ -4 & 4\end{bmatrix}=\begin{bmatrix}8\times2+5\times(-4) & 8\times\left(-\frac{5}{2}\right)+5\times4\\ 6\times2+4\times(-4) & 6\times\left(-\frac{5}{2}\right)+4\times4\end{bmatrix}$

$$=\begin{bmatrix}16-20 & -20+20\\ 12-16 & -15+16\end{bmatrix}=\begin{bmatrix}-4 & 0\\ -4 & 1\end{bmatrix}$$

(e) $\begin{bmatrix}-1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Solution: $\begin{bmatrix}-1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}(-1)\times0+2\times0 & (-1)\times0+2\times0\\ 1\times0+3\times0 & 1\times0+3\times0\end{bmatrix}=\begin{bmatrix}0+0 & 0+0\\ 0+0 & 0+0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

5. **Let** $A=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}, B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$ **and** $C=\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$**. Verify whether**

(i) $AB = BA$

Solution: L. H. S. = AB

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}=\begin{bmatrix}(-1)\times1+3\times(-3) & (-1)\times2+3\times(-5)\\ 2\times1+0\times(-3) & 2\times2+0\times(-5)\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4+0\end{bmatrix}=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix} \text{—(1)}$$

R.H.S. = BA

$$=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}=\begin{bmatrix}1\times(-1)+2\times2 & 1\times3+2\times0\\ (-3)\times(-1)+(-5)\times2 & (-3)\times3+(-5)\times0\end{bmatrix}$$

$$=\begin{bmatrix}-1+4 & 3+0\\ 3-10 & -9+0\end{bmatrix}=\begin{bmatrix}3 & 3\\ -7 & -9\end{bmatrix} \text{—(2)}$$

From (1) and (2), we have : $AB \neq BA$

(ii) $A(BC) = (AB)C$

Solution: L.H.S.= A (BC)

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}\right)=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1\times2+2\times1 & 1\times1+2\times3\\ (-3)\times2+(-5)\times1 & (-3)\times1+(-5)\times3\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}2+2 & 1+6\\ -6-5 & -3-15\end{bmatrix}=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}4 & 7\\ -11 & -18\end{bmatrix}$$

$$\begin{bmatrix}(-1)\times4+3\times(-11) & (-1)\times7+3\times(-18)\\ 2\times4+0\times(-11) & 2\times7+0\times(-18)\end{bmatrix}=\begin{bmatrix}-4-33 & -7-54\\ 8+0 & 14+0\end{bmatrix}=\begin{bmatrix}-37 & -61\\ 8 & 14\end{bmatrix} \text{—(1)}$$

R.H.S. = (AB) C

$$=\left(\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}\right)\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}=\begin{bmatrix}(-1)\times1+3\times(-3) & (-1)\times2+3\times(-5)\\ 2\times1+0\times(-3) & 2\times2+0\times(-5)\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4+0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}(-10)\times2+(-17)\times1 & (-10)\times1+(-17)\times3\\ 2\times2+4\times1 & 2\times1+4\times3\end{bmatrix}=\begin{bmatrix}-20-17 & -10-51\\ 4+4 & 2+12\end{bmatrix}=\begin{bmatrix}-37 & -61\\ 8 & 14\end{bmatrix} \text{(1)}$$

From (1) and (2), we have : $A(BC) = (AB)C$

(iii) A (B + C) = AB + AC

Solution: L.H.S. = A (B + C)

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}+\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}\right)=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1+2 & 2+1\\ -3+1 & -5+3\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}=\begin{bmatrix}3 & 3\\ -2 & -2\end{bmatrix}=\begin{bmatrix}(-1)\times3+3\times(-2) & (-1)\times3+3\times(-2)\\ 2\times3+0\times(-2) & 2\times3+0\times(-2)\end{bmatrix}$$

$$=\begin{bmatrix}-3-6 & -3-6\\ 6-0 & 6-0\end{bmatrix}=\begin{bmatrix}-9 & -9\\ 6 & 6\end{bmatrix}$$

R.H.S. = AB + AC

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}+\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}(-1)\times1+3\times(-3) & (-1)\times2+3\times(-5)\\ 2\times1+0\times(-3) & 2\times2+0\times(-5)\end{bmatrix}+\begin{bmatrix}(-1)\times2+3\times1 & (-1)\times1+3\times3\\ 2\times2+0\times1 & 2\times1+0\times3\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4+0\end{bmatrix}+\begin{bmatrix}-2+3 & -1+9\\ 4+0 & 2+0\end{bmatrix}$$

$$=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}+\begin{bmatrix}1 & 8\\ 4 & 2\end{bmatrix}=\begin{bmatrix}-10+1 & -17+8\\ 2+4 & 4+2\end{bmatrix}=\begin{bmatrix}-9 & -9\\ 6 & 6\end{bmatrix}\text{——(2)}$$

From (1) and (2), we have : A (B + C) = AB + AC

(iv) A (B – C) = AB – AC.

Solution: L. H. S. = A (B – C)

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}-\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}\right)=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1-2 & 2-1\\ -3-1 & -5-3\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}-1 & 1\\ -4 & -8\end{bmatrix}=\begin{bmatrix}(-1)\times(-1)+3\times(-4) & (-1)\times1+3\times(-8)\\ 2\times(-1)+0\times(-4) & 2\times1+0\times(-8)\end{bmatrix}=\begin{bmatrix}1-12 & -1-24\\ -2-0 & 2-0\end{bmatrix}$$

$$=\begin{bmatrix}-11 & -25\\ -2 & 2\end{bmatrix}\text{——(1)}$$

R.H.S.= AB – AC

$$=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}-\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}(-1)\times1+3\times(-3) & (-1)\times2+3\times(-5)\\ 2\times1+0\times(-3) & 2\times2+0\times(-5)\end{bmatrix}-\begin{bmatrix}(-1)\times2+3\times1 & (-1)\times1+3\times3\\ 2\times2+0\times1 & 2\times1+0\times3\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4+0\end{bmatrix}-\begin{bmatrix}-2+3 & -1+9\\ 4+0 & 2+0\end{bmatrix}=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}-\begin{bmatrix}1 & 8\\ 4 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-10-1 & -17-8\\ 2-4 & 4-2\end{bmatrix}=\begin{bmatrix}-11 & -25\\ -2 & 2\end{bmatrix}\text{——(2)}$$

From (1) and (2), we have : A (B – C) = AB – AC.

6. For the matrices $A=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}, B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}, C=\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}$. Verify that

(i) $(AB)^t = B^t A^t$

Solution: L.H.S. $= (AB)^t$

Now AB $=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}=\begin{bmatrix}(-1)\times1+3\times(-3) & (-1)\times2+3(-5)\\ 2\times1+0\times(-3) & 2\times2+0\times(-5)\end{bmatrix}$

$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4-0\end{bmatrix}=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}$

So, $(AB)^t =\begin{bmatrix}-10 & 2\\ -17 & 4\end{bmatrix}$ ——(1)

R.H.S. $= B^t A^t$

Now $A=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\Rightarrow A^t=\begin{bmatrix}-1 & 2\\ 3 & 0\end{bmatrix}$

$B=\begin{bmatrix}+1 & 2\\ -3 & -5\end{bmatrix}\Rightarrow B^t=\begin{bmatrix}+1 & -3\\ 2 & -5\end{bmatrix}$

So, $B^t A^t =\begin{bmatrix}+1 & -3\\ 2 & -5\end{bmatrix}\begin{bmatrix}-1 & 2\\ 3 & 0\end{bmatrix}=\begin{bmatrix}+1\times(-1)+(-3)\times3 & 1\times2+(-3)\times0\\ 2\times(-1)+(-5)\times3 & 2\times2+(-5)\times0\end{bmatrix}$

$=\begin{bmatrix}-1-9 & 2+0\\ -2-15 & 4-0\end{bmatrix}=\begin{bmatrix}-10 & 2\\ -17 & 4\end{bmatrix}$ ——(2)

From (1) and (2), we have $(AB)^t = B^t A^t$

(ii) $(BC)^t = C^t B^t$

Solution: L.H.S. $= (BC)^t$

Now BC $=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}=\begin{bmatrix}1\times(-2)+2\times3 & 1\times6+2\times(-9)\\ (-3)\times(-2)+(-5)\times3 & (-3)\times6+(-5)\times(-9)\end{bmatrix}$

$=\begin{bmatrix}-2+6 & 6-18\\ 6-15 & -18+45\end{bmatrix}=\begin{bmatrix}4 & -12\\ -9 & 27\end{bmatrix}$

So, $(BC)^t =\begin{bmatrix}4 & -9\\ -12 & 27\end{bmatrix}$ ——(1)

R.H.S. $= C^t B^t$

Now $C=\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}\Rightarrow C^t=\begin{bmatrix}-2 & 3\\ 6 & -9\end{bmatrix}$

$$B = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} \Rightarrow B^t = \begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix}$$

$$\text{So, } C^t\ B^t = \begin{bmatrix} -2 & 3 \\ 6 & -9 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} (-2)\times 1 + 3\times 2 & (-2)\times(-3) + 3\times(-5) \\ 6\times 1 + (-9)\times 2 & 6\times(-3) + (-9)\times(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -2+6 & 6-15 \\ 6-18 & -18+45 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ -12 & 27 \end{bmatrix} \text{—(2)}$$

From (1) and (2), we have: $(BC)^t = C^t B^t$

MULTIPLICATIVE INVERSE OF A MATRIX

Determinant of a 2 – by – 2 Matrix:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2 – by – 2 square matrix, the determinant of A, denoted by det. A or $|A|$ is defined as:

$$|A| = \det. A$$

$$= \det. \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\times d - b\times c = ad - bc$$

For example, (i) Let $B = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

Then $|B| = \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}$

$= (1)(3) - (-2)(1) \quad = 3 + 2 = 5$

(ii) If $M = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ Then $|M| = \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix}$

$= (2)(3) - (1)(6) = 6 - 6 = 0$

Singular and Non- Singular Matrix:

Singular Matrix: A square matrix A is called singular if the determinant of A is equal to zero. i.e., $|A| = 0$

For example, $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $|A| = (1)(0) - (2)(0) = 0 - 0 = 0$

Hence A is singular matrix.

Non- Singular Matrix: A square matrix A is called non- singular if the determinant of A is not equal to zero, i.e., $|A| \neq 0$

For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $|A| = (1)(2) - (0)(1) = 2 - 0 = 2$

Hence A is non- singular matrix.

Adjoint of a Matrix:

Adjoint of a square matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is obtained by interchanging the diagonal entries and changing the sign of other entries. It is denoted by Adj A.

i.e., $\text{Adj } A=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

For example, (i) If $A=\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, then $\text{Adj } A=\begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$

(ii) If $B=\begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}$, then $\text{Adj } B=\begin{bmatrix} -4 & 1 \\ -3 & 2 \end{bmatrix}$

Multiplicative Inverse of a Non- Singular Matrix:

Let A and B are two non- singular matrices of same order. Then A and B are said to be multiplicative inverse of each other, if AB = BA = I

The inverse of A is denoted by A^{-1}, thus

$A A^{-1} = A^{-1} A = I$

Remember: Inverse of a matrix is possible only if matrix is non- Singular.

Inverse of a Matrix using Adjoint:

Let $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix. To find the inverse of M, i.e., M^{-1}, first we find the determinant as inverse is possible only of a non- singular matrix.

$|M|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc\neq 0$

and $\text{Adj } M=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ Then $M^{-1}=\dfrac{\text{Adj}M}{|M|}=\dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$

For example, let $A=\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$ Then $|A|=\begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix}=(2)(-3)-(1)(-1)$

$=-6+1=-5\neq 0$

$\text{Adj } A=\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$ Thus $A^{-1}=\dfrac{1}{|A|}\text{Adj}A$

$=\dfrac{1}{-5}\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$

Verification of $(AB)^{-1} = B^{-1} A^{-1}$

Let $A=\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$ $|A|=\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix}$ $|B|=\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix}$

$=(3)(0)-(-1)(1)$ $\quad=(0)(2)-(-1)(3)$

$= 0 + 1$ $\qquad\qquad = 0 + 3$

$= 1 \neq 0$ $\qquad\qquad = 3 \neq 0$

Therefore, A and B are invertible. i.e., their inverses exists.

Then, to verify the law of inverse of the product, take.

$$AB = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3\times0+1\times3 & 3\times(-1)+1\times2 \\ (-1)\times0+0\times3 & (-1)\times(-1)+0\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+3 & -3+2 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = (3)(1) - (-1)(0) = 3 + 0 = 3 \neq 0$$

$$\text{Adj}\,(AB) = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\text{L.H.S} = (AB)^{-1} = \frac{1}{|AB|}\text{Adj}(AB) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \text{——(1)}$$

And R.H.S. $= B^{-1}A^{-1}$

$$B^{-1} = \frac{1}{|B|}\text{Adj }B = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj }A = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

Now $B^{-1}A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2\times0+1\times1 & 2\times(-1)+1\times3 \\ (-3)\times+0\times1 & (-3)\times(-1)+0\times3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0+1 & -2+3 \\ 0+0 & 3+0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \text{——(2)}$$

From (1) and (2), we have: $(AB)^{-1} = B^{-1}A^{-1}$

Solved Exercise 1.5

1. Find the determinant of the following matrices.

(i) $A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$

Solution: $|A| = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = (-1)(0) - (1)(2) = 0 - 2 = -2$

(ii) $B = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$

Solution: $|B| = \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = (1)(-2) - (3)(2) = -2 - 6 = -8$

(iii) $C=\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$

Solution: $|C|=\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix}=(3)(2)-(3)(2)=6-6=0$

(iv) $D=\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

Solution: $|D|=\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}=(3)(4)-(2)(1)=12-2=10$

2. Find which of the following matrices are singular or non- singular?

(i) $A=\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

Solution: $|A|=\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix}=(3)(4)-(2)(6)=12-12=0$

Hence matrix A is singular.

(ii) $B=\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Solution: $|B|=\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}=(4)(2)-(1)(3)=8-3=5\neq 0$

Hence matrix B is non-singular.

(iii) $C=\begin{bmatrix} 7 & -9 \\ 3 & 5 \end{bmatrix}$

Solution: $|C|=\begin{vmatrix} 7 & -9 \\ 3 & 5 \end{vmatrix}=(7)(5)-(3)(-9)=35+27=62\neq 0$

Hence matrix C is non- singular.

(iv) $D=\begin{bmatrix} 5 & -10 \\ -2 & 4 \end{bmatrix}$

Solution: $|D|=\begin{vmatrix} 5 & -10 \\ -2 & 4 \end{vmatrix}=(5)(4)-(-10)(-2)=20-20=0$

Hence matrix D is singular.

3. Find the multiplicative inverse (if it exists) of each.

(i) $A=\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

Solution: $A=\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}\Rightarrow |A|=\begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix}=(-1)(0)-(3)(2)=0-6=-6$

$$\text{Adj } A=\begin{bmatrix} 0 & -3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\text{Adj}A=\frac{1}{-6}\begin{bmatrix} 0 & -3 \\ -2 & -1 \end{bmatrix}=\begin{bmatrix} \frac{0}{-6} & -\frac{3}{-6} \\ \frac{-2}{-6} & \frac{-1}{-6} \end{bmatrix}=\begin{bmatrix} 0 & +\frac{1}{2} \\ +\frac{1}{3} & +\frac{1}{6} \end{bmatrix}$$

(ii) $B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$

Solution: $B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix} \Rightarrow |B| = \begin{vmatrix}1 & 2\\ -3 & -5\end{vmatrix} = (1)(-5)-(2)(-3) = -5+6 = 1$

$\text{Adj } B = \begin{bmatrix}-5 & -2\\ 3 & 1\end{bmatrix}$, $B^{-1} = \frac{1}{|B|}\text{Adj}B = \frac{1}{1}\begin{bmatrix}-5 & -2\\ 3 & 1\end{bmatrix} = \begin{bmatrix}-5 & -2\\ 3 & 1\end{bmatrix}$

(iii) $C=\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}$

Solution: $C=\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix} \Rightarrow |C| = \begin{vmatrix}-2 & 6\\ 3 & -9\end{vmatrix} = (-2)(-9)-(6)(3) = 18-18 = 0$

Hence C^{-1} does not exist.

(iv) $D=\begin{bmatrix}\frac{1}{2} & \frac{3}{4}\\ 1 & 2\end{bmatrix}$

Solution: $D=\begin{bmatrix}\frac{1}{2} & \frac{3}{4}\\ 1 & 2\end{bmatrix} \Rightarrow |D| = \begin{vmatrix}\frac{1}{2} & \frac{3}{4}\\ 1 & 2\end{vmatrix} = \left(\frac{1}{2}\right)(2)-(1)\left(\frac{3}{4}\right) = 1-\frac{3}{4} = \frac{1}{4}$

$\text{Adj } D = \begin{bmatrix}2 & -\frac{3}{4}\\ -1 & \frac{1}{2}\end{bmatrix}$, $= D^{-1} = \frac{1}{|D|}\text{Adj } D = \frac{1}{\frac{1}{4}}\begin{bmatrix}2 & -\frac{3}{4}\\ -1 & \frac{1}{2}\end{bmatrix} = 4\begin{bmatrix}2 & -\frac{3}{4}\\ -1 & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}8 & -3\\ -4 & 2\end{bmatrix}$

4. **If** $A=\begin{bmatrix}1 & 2\\ 4 & 6\end{bmatrix}$ **and** $B=\begin{bmatrix}3 & -1\\ 2 & -2\end{bmatrix}$**, then**

(i) $A\,(\text{Adj } A) = (\text{Adj } A)\,A = (\det A)\,I$

Solution: $A=\begin{bmatrix}1 & 2\\ 4 & 6\end{bmatrix} \Rightarrow |A| = \begin{vmatrix}1 & 2\\ 4 & 6\end{vmatrix} = (1)(6)-(2)(4) = 6-8 = -2$

$\text{Adj } A = \begin{bmatrix}6 & -2\\ -4 & 1\end{bmatrix}$

Now $A\,(\text{Adj}A) = \begin{bmatrix}1 & 2\\ 4 & 6\end{bmatrix}\begin{bmatrix}6 & -2\\ -4 & 1\end{bmatrix} = \begin{bmatrix}1\times6+2\times(-4) & 1\times(-2)+2\times1\\ 4\times6+6\times(-4) & 4\times(-2)+6\times1\end{bmatrix}$

$= \begin{bmatrix}6-8 & -2+2\\ 24-24 & -8+6\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$ —— (1)

$(\text{Adj } A)\ A = \begin{bmatrix}6 & -2\\ -4 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 4 & 6\end{bmatrix} = \begin{bmatrix}6\times1+(-2)\times4 & 6\times2+(-2)\times6\\ (-4)\times1+1\times4 & (-4)\times2+1\times6\end{bmatrix}$

$= \begin{bmatrix}6-8 & 12-12\\ -4+4 & -8+6\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$ —— (2)

$(\det A)\,I = -2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$ —— (3)

From (1), (2) and (3), we have

$$A\,(\text{Adj } A) = (\text{Adj } A)\,A = (\det A)\,I$$

(ii) $\mathbf{B\,B^{-1} = I = B^{-1}\,B}$

Solution: $B = \begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix} \Rightarrow |B| = \begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix} = (3)(-2) - (-1)(2) = -6 + 2 = -4$

$$\text{Adj } B = \begin{bmatrix} -2 & 1 \\ -2 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\text{Adj } B = \frac{1}{-4}\begin{bmatrix} -2 & 1 \\ -2 & 3 \end{bmatrix}$$

Now $B\,B^{-1}$

$$\begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix}\frac{1}{-4}\begin{bmatrix} -2 & 1 \\ -2 & 3 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -2 & 3 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 3\times(-2)+(-1)\times(-2) & 3\times1+(-1)\times3 \\ 2\times(-2)+(-2)\times(-2) & 2\times1+(-2)\times3 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -6+2 & 3-3 \\ -4+4 & 2-6 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} \frac{-4}{-4} & \frac{0}{-4} \\ \frac{0}{-4} & \frac{-4}{-4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now, $B^{-1}\,B = -\frac{1}{4}\begin{bmatrix} -2 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} (-2)\times3+1\times2 & (-2)\times(-1)+1\times(-2) \\ (-2)\times3+3\times2 & (-2)\times(-1)+3\times(-2) \end{bmatrix}$

$$= -\frac{1}{4}\begin{bmatrix} -6+2 & 2-2 \\ -6+6 & 2-6 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} \frac{-4}{-4} & \frac{0}{-4} \\ \frac{0}{-4} & \frac{-4}{-4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $B\,B^{-1} = B^{-1}\,B = I$

5. Determine whether the given matrices are multiplicative inverse of each other.

(i) $\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ **and** $\begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$

Solution: $\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3\times7+5\times(-4) & 3\times(-5)+5\times3 \\ 4\times7+7\times(-4) & 4\times(-5)+7\times3 \end{bmatrix}$

$$= \begin{bmatrix} 21-20 & -15+15 \\ 28-28 & -20+21 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Yes, the given matrices are multiplicative inverse of each other.

(ii) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ **and** $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1\times(-3)+2\times2 & 1\times(+2)+2\times(-1) \\ 2\times(-3)+3\times2 & 2\times(+2)+3\times(-1) \end{bmatrix} = \begin{bmatrix} -3+4 & +2-2 \\ -6+6 & +4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Yes, the given matrices are multiplicative inverse of each other.

6. If $A=\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}, B=\begin{bmatrix}-4 & -2\\ 1 & -1\end{bmatrix}, D=\begin{bmatrix}3 & 1\\ -2 & 2\end{bmatrix}$, then verify that

(i) $(AB)^{-1}=B^{-1}A^{-1}$

Solution: L.H.S.= $(AB)^{-1}$

$$(AB)=\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}-4 & -2\\ +1 & -1\end{bmatrix}=\begin{bmatrix}4\times(-4)+0\times(+1) & 4\times(-2)+0\times(-1)\\ (-1)\times(-4)+2\times(+1) & (-1)\times(-2)+2\times(-1)\end{bmatrix}$$

$$=\begin{bmatrix}-16+0 & -8+0\\ +4+2 & 2-2\end{bmatrix}=\begin{bmatrix}-16 & -8\\ 6 & 0\end{bmatrix}$$

$$|AB|=\begin{vmatrix}-16 & -8\\ 6 & 0\end{vmatrix}=(-16)(0)-(-8)(6)=0+48=48$$

$$\text{Adj }(AB)=\begin{bmatrix}0 & 8\\ -6 & -16\end{bmatrix}$$

$$(AB)^{-1}=\frac{1}{|AB|}\text{Adj }(AB)=\frac{1}{48}\begin{bmatrix}0 & 8\\ -6 & -16\end{bmatrix}=\begin{bmatrix}\frac{0}{48} & \frac{8}{48}\\ \frac{-6}{48} & \frac{-16}{48}\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{6}\\ -\frac{1}{8} & \frac{-1}{3}\end{bmatrix}\text{——(1)}$$

R.H.S.= $B^{-1}A^{-1}$

$$B=\begin{bmatrix}-4 & -2\\ 1 & -1\end{bmatrix}\Rightarrow|B|=\begin{vmatrix}-4 & -2\\ 1 & -1\end{vmatrix}=(-4)(-1)-(1)(-2)=4+2=6$$

$$\text{Adj }B=\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}\Rightarrow B^{-1}=\frac{1}{|B|}\text{Adj }B=\frac{1}{6}\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}$$

$$A=\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\Rightarrow|A|=\begin{vmatrix}4 & 0\\ -1 & 2\end{vmatrix}=(4)(2)-(0)(-1)=8-0=8$$

$$\text{Adj }A=\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}\Rightarrow A^{-1}=\frac{1}{|A|}\text{Adj }A=\frac{1}{8}\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}$$

$$B^{-1}A^{-1}=\frac{1}{6}\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}\frac{1}{8}\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}=\frac{1}{48}\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}$$

$$=\frac{1}{48}\begin{bmatrix}(-1)\times2+2\times1 & (-1)\times0+2\times4\\ (-1)\times2+(-4)\times1 & (-1)\times0+(-4)\times4\end{bmatrix}=\frac{1}{48}\begin{bmatrix}-2+2 & 0+8\\ -2-4 & 0-16\end{bmatrix}$$

$$=\frac{1}{48}\begin{bmatrix}-2+2 & 0+8\\ -2-4 & 0-16\end{bmatrix}=\begin{bmatrix}\frac{0}{48} & \frac{8}{48}\\ \frac{-6}{48} & \frac{-16}{48}\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{6}\\ -\frac{1}{8} & -\frac{1}{3}\end{bmatrix}\text{——(2)}$$

From (1) and (2), we have: $(AB)^{-1}=B^{-1}A^{-1}$

(ii) $(DA)^{-1}=A^{-1}D^{-1}$

Solution: L.H.S.= $(DA)^{-1}$

$$\text{Now }DA=\begin{bmatrix}3 & 1\\ -2 & 2\end{bmatrix}\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}=\begin{bmatrix}3\times4+1\times(-1) & 3\times0+1\times2\\ (-2)\times4+2\times(-1) & (-2)\times0+2\times2\end{bmatrix}$$

$$=\begin{bmatrix}12-1 & 0+2\\ -8-2 & 0+4\end{bmatrix}=\begin{bmatrix}11 & 2\\ -10 & 4\end{bmatrix}\Rightarrow |DA|=\begin{vmatrix}11 & 2\\ -10 & 4\end{vmatrix}=(11)(4)-(2)(-10)=44+20=64$$

$$\text{Adj } DA=\begin{bmatrix}4 & -2\\ 10 & 11\end{bmatrix}\Rightarrow (DA)^{-1}=\frac{1}{|DA|}\text{Adj } DA$$

$$(DA)^{-1}=\frac{1}{64}\begin{bmatrix}4 & -2\\ 10 & 11\end{bmatrix}=\begin{bmatrix}\frac{4}{64} & \frac{-2}{64}\\ \frac{10}{64} & \frac{11}{64}\end{bmatrix}=\begin{bmatrix}\frac{1}{16} & -\frac{1}{32}\\ \frac{5}{32} & \frac{11}{64}\end{bmatrix}\text{——}(1)$$

R.H.S. $= A^{-1}D^{-1}$

$$A=\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\Rightarrow |A|=\begin{vmatrix}4 & 0\\ -1 & 2\end{vmatrix}=(4)(2)-(-1)(0)=8+0=8$$

$$\text{Adj } A=\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}\Rightarrow A^{-1}=\frac{1}{|A|}\text{Adj } A=\frac{1}{8}\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}$$

$$D=\begin{bmatrix}3 & 1\\ -2 & 2\end{bmatrix}$$

$$|D|=\begin{vmatrix}3 & 1\\ -2 & 2\end{vmatrix}=(3)(2)-(1)(-2)=6+2=8$$

$$\text{Adj } D=\begin{bmatrix}2 & -1\\ 2 & 3\end{bmatrix}\Rightarrow D^{-1}=\frac{1}{|D|}\text{Adj } D=\frac{1}{8}\begin{bmatrix}2 & -1\\ 2 & 3\end{bmatrix}$$

Now $A^{-1}D^{-1}=\frac{1}{8}\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}\frac{1}{8}\begin{bmatrix}2 & -1\\ 2 & 3\end{bmatrix}=\frac{1}{64}\begin{bmatrix}2\times 2+0\times 2 & 2\times(-1)+0\times 3\\ 1\times 2+4\times 2 & 1\times(-1)+4\times 3\end{bmatrix}$

$$=\frac{1}{64}\begin{bmatrix}4+0 & -2+0\\ 2+8 & -1+12\end{bmatrix}=\frac{1}{64}\begin{bmatrix}4 & -2\\ 10 & 11\end{bmatrix}=\begin{bmatrix}\frac{4}{64} & \frac{-2}{64}\\ \frac{10}{64} & \frac{11}{64}\end{bmatrix}=\begin{bmatrix}\frac{1}{16} & -\frac{1}{32}\\ \frac{5}{32} & \frac{11}{64}\end{bmatrix}\text{——}(2)$$

From (1) and (2), we have: $(DA)^{-1}=A^{-1}\ D^{-1}$

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

System of two linear equations in two variables in general form is given as:

$ax + by = m \qquad cx + dy = n$

Where a, b, c, d, m and n are real numbers.

This system is also called simultaneous linear equations. We discuss here the following methods of solution.

(i) Matrix inversion method. **(ii)** Cramer's rule.

(i) Matrix inversion method.

Consider the system of linear equations.

$ax + by = m \qquad cx + dy = n$

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$

Or $A X = B \Rightarrow X = A^{-1} B$

Where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} m \\ n \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|} \text{Adj } A$

And $|A| = ad - bc, \text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} dm - bn \\ -cm + an \end{bmatrix} = \begin{bmatrix} \frac{dm - bn}{ad - bc} \\ \frac{an - cm}{ad - bc} \end{bmatrix}$$

$$\Rightarrow x = \frac{dm - bn}{ad - bc}, y = \frac{an - cm}{ad - bc}$$

(ii) Cramer's rule.

Consider the following system of linear equations.

$ax + by = m \qquad cx + dy = n$

We know that, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} m \\ n \end{bmatrix}$

So, $AX = B$

$X = A^{-1} B, \; X = \frac{1}{|A|} \text{Adj } A \times B$

Or $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} dm - bn \\ -cm + an \end{bmatrix} \Rightarrow x = \frac{dm - bn}{|A|}, y = \frac{an - cm}{|A|} = \frac{|A_x|}{|A|} = \frac{|A_y|}{|A|}$

Where $|A_x| \begin{vmatrix} m & b \\ n & d \end{vmatrix}$ and $|A_y| = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$

Example –1: Solve the following system by using matrices inversion method.

$4x - 2y = 8, \qquad 3x + y = -4$

Solution: $4x - 2y = 8, \qquad 3x + y = -4$

In matrix form $\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$

Here $A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$

So, $AX = B \qquad X = A^{-1} B$

And $A^{-1} = \frac{1}{|A|}\text{Adj } A$

Now $|A| = \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} = (4)(1) - (-2)(3) = 4 + 6 = 10 \neq 0$

$$\text{Adj } A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = A^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

So, $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1\times 8 + 2\times(-4) \\ (-3)\times 8 + 4\times(-4) \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8-8 \\ -24-16 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} \frac{0}{10} \\ \frac{-40}{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} \Rightarrow X = 0, y = -4$$

Example 2: Solve the following system of linear equations by using cramer's rule.
$3x - 2y = 1 \qquad -2x + 3y = 2$

Solution: $3x - 2y = 1 \qquad -2x + 3y = 2$

We have $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}, A_x = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, A_y = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix}, |A_x| = \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}, |A_y| = \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix}$$

$$|A| = (3)(3) - (-2)(-2), |A_x| = (1)(3) - (2)(-2), |A_y| = (3)(2) - (1)(-2)$$
$$= 9 - 4 = 5 \qquad = 3 + 4 = 7 \qquad = 6 + 2 = 8$$

Now $x = \frac{|A_x|}{|A|}$ And $y = \frac{|A_y|}{|A|}$

$= \frac{7}{5} \qquad = \frac{8}{5}$

Example- 3: If length of a rectangle is 6cm less than three times its width. The perimeter of the rectangle is 140 feet. Find the dimensions of the rectangle. (by using matrix inversion method).

Solution: If width of the rectangle is x cm, then length of the rectangle is:

$y = 3x - 6$

Or $3x - y = 6$ ______ (1)

Now Perimeter $= 2x + 2y$

$140 = 2(x + y) \quad \Rightarrow \quad x + y = 70$ ____ (2)

In matrix form, we have

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 6 \\ 70 \end{bmatrix}$$

So, $AX = B \qquad X = A^{-1}B$

And $A^{-1} = \frac{1}{|A|}\text{ Adj } A$

$$|A|=\begin{vmatrix}3 & -1\\1 & 1\end{vmatrix}=(3)(1)-(-1)(1)=3+1=4$$

$$\text{Adj } A=\begin{bmatrix}1 & 1\\-1 & 3\end{bmatrix}, A^{-1}=\frac{1}{4}\begin{bmatrix}1 & 1\\-1 & 3\end{bmatrix}$$

Now $$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1 & 1\\-1 & 3\end{bmatrix}\begin{bmatrix}6\\70\end{bmatrix}=\frac{1}{4}\begin{bmatrix}6+70\\-6+210\end{bmatrix}=\frac{1}{4}\begin{bmatrix}76\\204\end{bmatrix}$$

$$=\begin{bmatrix}\frac{76}{4}\\\frac{204}{4}\end{bmatrix}=\begin{bmatrix}19\\51\end{bmatrix}\Rightarrow x=19, y=51.$$

Width of rectangle $= x = 19$ cm
Length of rectangle $= y = 3x-6$
$= 3(19)-6 = 57-6 = 51$ cm

Verification:
Perimeter $= 2x+2y = (19)+2(51) = 38+102 = 140$ cm
$y = 3x-6 = 3(19)-6 = 57-6 = 51$ cm

Solved Exercise 1.6

1. Use matrices, if possible, to solve the following systems of linear equations by:
(i) the matrix inversion method. (ii) the Cramer's rule.

(i) $2x-2y=4$ $3x+2y=6$

Solution: (i) The matrix inversion method:

$2x-2y=4 \quad 3x+2y=6$

In matrix form: $\begin{bmatrix}2 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$ Here, $A=\begin{bmatrix}2 & -2\\3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}, B=\begin{bmatrix}4\\6\end{bmatrix}$

So, $AX=B \Rightarrow X=A^{-1}B$

Now $A^{-1}=\frac{1}{|A|}\text{Adj } A$

$$|A|=\begin{vmatrix}2 & -2\\3 & 2\end{vmatrix}=(2)(2)-(3)(-2)=4+6=10$$

$$\text{Adj}A=\begin{bmatrix}2 & 2\\-3 & 2\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{10}\begin{bmatrix}2 & 2\\-3 & 2\end{bmatrix}$$

So, $$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{10}\begin{bmatrix}2 & 2\\-3 & 2\end{bmatrix}\begin{bmatrix}4\\6\end{bmatrix}=\frac{1}{10}\begin{bmatrix}2\times4+2\times6\\(-3)\times4+2\times6\end{bmatrix}=\frac{1}{10}\begin{bmatrix}8+12\\-12+12\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\0\end{bmatrix}=\begin{bmatrix}\frac{20}{10}\\\frac{0}{10}\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$$

$\Rightarrow x=2, y=0$

(ii) The Cramer's rule:

$2x-2y=4 \quad 3x+2y=6$

We have $A = \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & -2 \\ 3 & 2 \end{vmatrix} = (2)(2)-(-2)(3) = 4+6 = 10$

$A_x = \begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix} \Rightarrow |A_x| = \begin{vmatrix} 4 & -2 \\ 6 & 2 \end{vmatrix} = (4)(2)-(-2)(6) = 8+12 = 20$

$A_y = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \Rightarrow |A_y| = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = (2)(6)-(3)(4) = 12-12 = 0$

Now $x = \dfrac{|A_x|}{|A|} = \dfrac{20}{10} = 2$

And $y = \dfrac{|A_y|}{|A|} = \dfrac{0}{10} = 0$

(ii) $2x + y = 3 \qquad 6x + 5y = 1$

Solution: (i) The matrix inversion method:

$2x + y = 3 \qquad 6x + 5y = 1$

In matrix form: $\begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ Here, $A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

So, $AX = B \Rightarrow X = A^{-1}B$

Now, $A^{-1} = \dfrac{1}{|A|}\text{Adj } A$

$|A| = \begin{vmatrix} 2 & 1 \\ 6 & 5 \end{vmatrix} = (2)(5)-(1)(6) = 10-6 = 4$

$\text{Adj } A = \begin{bmatrix} 5 & -1 \\ -6 & 2 \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{4}\begin{bmatrix} 5 & -1 \\ -6 & 2 \end{bmatrix}$

So, $\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{4}\begin{bmatrix} 5 & -1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \dfrac{1}{4}\begin{bmatrix} 5\times 3+(-1)\times 1 \\ (-6)\times 3+2\times 1 \end{bmatrix} = \dfrac{1}{4}\begin{bmatrix} 15-1 \\ -18+2 \end{bmatrix} = \dfrac{1}{4}\begin{bmatrix} 14 \\ -16 \end{bmatrix} = \begin{bmatrix} \frac{14}{4} \\ \frac{-16}{4} \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ -4 \end{bmatrix}$

$\Rightarrow \quad x = \dfrac{7}{2}, y = -4$

(ii) The Cramer's rule:

$2x + y = 3 \qquad 6x + 5y = 1$

Here $A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}, A_x = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}, A_y = \begin{bmatrix} 2 & 3 \\ 6 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 1 \\ 6 & 5 \end{vmatrix}, |A_x| = \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix}, |A_y| = \begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix}$

$|A| = (2)(5)-(1)(6), \quad |A_x| = (3)(5)-(1)(1), \quad |A_y| = (2)(1)-(6)(3)$

$= 10-6 = 4 \qquad = 15-1 = 14 \qquad = 2-18 = -16$

Now $x = \dfrac{|A_x|}{|A|} \qquad y = \dfrac{|A_y|}{|A|}$

$=\frac{14}{4}=\frac{7}{2}$ $\qquad\qquad$ $=\frac{-16}{4}=-4$

(iii) $\mathbf{4x+2y=8}$
$\mathbf{3x-y=-1}$

Solution: (i) The matrix inversion method:

$4x+2y=8 \qquad 3x-y=-1$

In matrix form: $\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-1\end{bmatrix}$ Here, $A=\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}, B=\begin{bmatrix}8\\-1\end{bmatrix}$

So, $AX=B$

And $A^{-1}=\frac{1}{|A|}\text{Adj }A$

$$|A|=\begin{vmatrix}4 & 2\\3 & -1\end{vmatrix}=(4)(-1)-(2)(3)=-4-6=-10$$

$$\text{Adj }A=\begin{bmatrix}-1 & -2\\-3 & 4\end{bmatrix}\Rightarrow A^{-1}=\frac{1}{-10}\begin{bmatrix}-1 & -2\\-3 & 4\end{bmatrix}$$

So, $\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{-10}\begin{bmatrix}-1 & -2\\-3 & 4\end{bmatrix}\begin{bmatrix}8\\-1\end{bmatrix}=\frac{1}{-10}\begin{bmatrix}(-1)\times 8+(-2)\times(-1)\\(-3)\times 8+4\times(-1)\end{bmatrix}$

$$=\frac{1}{-10}\begin{bmatrix}-8+2\\-24-4\end{bmatrix}=\frac{1}{-10}\begin{bmatrix}-6\\-28\end{bmatrix}=\begin{bmatrix}\frac{-6}{-10}\\\frac{-28}{-10}\end{bmatrix}=\begin{bmatrix}\frac{3}{5}\\\frac{14}{5}\end{bmatrix}$$

$\Rightarrow x=\frac{3}{5}, \qquad y=\frac{14}{5}$

(ii) **The Cramer's rule:**

$4x+2y=8$
$3x-y=-1$

Here $\quad A=\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix}, \quad A_x=\begin{bmatrix}8 & 2\\-1 & -1\end{bmatrix}, A_y=\begin{bmatrix}4 & 8\\3 & -1\end{bmatrix}$

$$|A|=\begin{vmatrix}4 & 2\\3 & -1\end{vmatrix}, |A_x|=\begin{vmatrix}8 & 2\\-1 & -1\end{vmatrix}, |A_y|=\begin{vmatrix}4 & 8\\3 & -1\end{vmatrix}$$

$|A|=(4)(-1)-(2)(3), |A_x|=(8)(-1)-(2)(-1), |A_y|=(4)(-1)-(8)(3)$

$=-4-6 \qquad\qquad =-8+2 \qquad\qquad =-4-24$

$=-10 \qquad\qquad =-6 \qquad\qquad =-28$

Now $x=\frac{|A_x|}{|A|}$ $\qquad\qquad$ $y=\frac{|A_y|}{|A|}$

$=\frac{-6}{-10}$ $\qquad\qquad$ $=\frac{-28}{-10}$

$=\frac{3}{5}$ $\qquad\qquad$ $=\frac{14}{5}$

(iv) $\mathbf{3x-2y=-6 \quad 5x-2y=-10}$

Solution: (i) The matrix inversion method:

$3x-2y=-6 \qquad 5x-2y=-10$

In matrix form: $\begin{bmatrix}3 & -2\\5 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-6\\-10\end{bmatrix}$ Here, $A=\begin{bmatrix}3 & -2\\5 & -2\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}, B=\begin{bmatrix}-6\\-10\end{bmatrix}$

$AX=B$

So, $X=A^{-1}B$

And $A^{-1}=\frac{1}{|A|}\text{Adj}A$

$$|A|=\begin{vmatrix}3 & -2\\5 & -2\end{vmatrix}=(3)(-2)-(5)(-2)=-6+10=4$$

$$\text{Adj }A=\begin{bmatrix}-2 & +2\\-5 & +3\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}-2 & +2\\-5 & +3\end{bmatrix}$$

So, $$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-2 & +2\\-5 & +3\end{bmatrix}\begin{bmatrix}-6\\-10\end{bmatrix}=\frac{1}{4}\begin{bmatrix}(-2)\times(-6)+2\times(-10)\\(-5)\times(-6)+3\times(-10)\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}12-20\\+30-30\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-8\\0\end{bmatrix}=\begin{bmatrix}\frac{-8}{4}\\\frac{0}{4}\end{bmatrix}=\begin{bmatrix}-2\\0\end{bmatrix}$$

$\Rightarrow x=-2, y=0$

(ii) The cramer's rule:

$3x-2y=-6$
$5x-2y=-10$

Here $A=\begin{bmatrix}3 & -2\\5 & -2\end{bmatrix}\Rightarrow \quad |A|=\begin{vmatrix}3 & -2\\5 & -2\end{vmatrix}=(3)(-2)-(5)(-2)=-6+10=4$

$A_x=\begin{bmatrix}-6 & -2\\-10 & -2\end{bmatrix}\Rightarrow \quad |A_x|=\begin{vmatrix}-6 & -2\\-10 & -2\end{vmatrix}=(-6)(-2)-(-10)(-2)=+12-20=-8$

$A_y=\begin{bmatrix}3 & -6\\5 & -10\end{bmatrix}\Rightarrow \quad |A_y|=\begin{vmatrix}3 & -6\\5 & -10\end{vmatrix}=(3)(-10)-(5)(-6)=-30+30=0$

Solution:

Now $x=\frac{|A_x|}{|A|}=\frac{-8}{4}=-2$ And $y=\frac{|A_y|}{|A|}=\frac{0}{4}=0$

(v) $\mathbf{3x-2y=4}$
$\mathbf{-6x+4y=7}$

Solution: (i) The matrix inversion method:

$3x-2y=4$
$-6x+4y=7$

In matrix form: $\begin{bmatrix}3 & -2\\ -6 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7\end{bmatrix}$ Here, $A=\begin{bmatrix}3 & -2\\ -6 & 4\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}, B=\begin{bmatrix}4\\ 7\end{bmatrix}$

So, $AX=B \Rightarrow X=A^{-1}B$

And $A^{-1}=\dfrac{1}{|A|}\text{Adj } A$

Now $|A|=\begin{vmatrix}3 & -2\\ -6 & 4\end{vmatrix}=(3)(4)-(-2)(-6)=12-12=0$

Hence Solution is not possible.

(vi) $\mathbf{4x+y=9}$
$\mathbf{-3x-y=-5}$

Solution: (i) The matrix inversion method:

$4x+y=9$
$-3x-y=-5$

In matrix form: $\begin{bmatrix}4 & 1\\ -3 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}9\\ -5\end{bmatrix}$ Here, $A=\begin{bmatrix}4 & 1\\ -3 & -1\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}, B=\begin{bmatrix}9\\ -5\end{bmatrix}$

So, $AX=B$
$\Rightarrow X=A^{-1}B$

And $A^{-1}=\dfrac{1}{|A|}\text{Adj}A$

$|A|=\begin{vmatrix}4 & 1\\ -3 & -1\end{vmatrix}=(4)(-1)-(-3)(1)=-4+3=-1$

$\text{Adj}A=\begin{bmatrix}-1 & -1\\ 3 & 4\end{bmatrix}$

$A^{-1}=\dfrac{1}{-1}\begin{bmatrix}-1 & -1\\ 3 & 4\end{bmatrix}=-\begin{bmatrix}-1 & -1\\ 3 & 4\end{bmatrix}=\begin{bmatrix}1 & 1\\ -3 & -4\end{bmatrix}$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1 & 1\\ -3 & -4\end{bmatrix}\begin{bmatrix}9\\ -5\end{bmatrix}=\begin{bmatrix}1\times 9+1\times(-5)\\ (-3)\times 9+(-4)\times(-5)\end{bmatrix}=\begin{bmatrix}9-5\\ -27+20\end{bmatrix}=\begin{bmatrix}4\\ -7\end{bmatrix}$$

$\Rightarrow \quad x=4, y=-7$

(ii) The cramer's rule:

$4x+y=9$
$-3x-y=-5$

Solution:

Here $A=\begin{bmatrix}4 & 1\\ -3 & -1\end{bmatrix} \Rightarrow |A|=\begin{vmatrix}4 & 1\\ -3 & -1\end{vmatrix}=(4)(-1)-(-3)(1)=-4+3=-1$

$A_x=\begin{bmatrix}9 & 1\\ -5 & -1\end{bmatrix} \Rightarrow |A_x|=\begin{vmatrix}9 & 1\\ -5 & -1\end{vmatrix}=(9)(-1)-(-5)(1)=-9+5=-4$

$$A_y = \begin{bmatrix} 4 & 9 \\ -3 & -5 \end{bmatrix} \Rightarrow |A_y| = \begin{vmatrix} 4 & 9 \\ -3 & -5 \end{vmatrix} = (4)(-5) - (9)(-3) = -20 + 27 = 7$$

Now $x = \dfrac{|A_x|}{|A|} = \dfrac{-4}{-1} = 4$

$y = \dfrac{|A_y|}{|A|} = \dfrac{7}{-1} = -7$

(vii) $\mathbf{2x - 2y = 4}$

$\mathbf{-5x - 2y = -10}$

Solution: (i) The matrix inversion method:

$$2x - 2y = 4$$
$$-5x - 2y = -10$$

In matrix form: $\begin{bmatrix} 2 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$ Here, $A = \begin{bmatrix} 2 & -2 \\ -5 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

So, $AX = B \Rightarrow X = A^{-1}\ B$

And $A^{-1} = \dfrac{1}{|A|}\text{Adj } A$

$$|A| = \begin{vmatrix} 2 & -2 \\ -5 & -2 \end{vmatrix} = (2)(-2) - (-2)(-5) = -4 - 10 = -14$$

$$\text{Adj } A = \begin{bmatrix} -2 & 2 \\ 5 & 2 \end{bmatrix} = A^{-1} = \frac{1}{-14}\begin{bmatrix} -2 & 2 \\ 5 & 2 \end{bmatrix}$$

So, $$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-14}\begin{bmatrix} -2 & 2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -10 \end{bmatrix} = \frac{1}{-14}\begin{bmatrix} (-2)\times 4 + 2\times(-10) \\ 5\times 4 + 2\times(-10) \end{bmatrix}$$

$$= \frac{1}{-14}\begin{bmatrix} -8-20 \\ 20-20 \end{bmatrix} = \frac{1}{-14}\begin{bmatrix} -28 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-28}{-14} \\ \frac{0}{-14} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x = 2, y = 0$

(ii) The cramer's rule:

$2x - 2y = 4$

$-5x - 2y = -10$

Solution:

Here $A = \begin{bmatrix} 2 & -2 \\ -5 & -2 \end{bmatrix} \Rightarrow \quad |A| = \begin{vmatrix} 2 & -2 \\ -5 & -2 \end{vmatrix} = (2)(-2) - (-5)(-2) = -4 - 10 = -14$

$A_x = \begin{bmatrix} 4 & -2 \\ -10 & -2 \end{bmatrix} \Rightarrow \quad |A_x| = \begin{vmatrix} 4 & -2 \\ -10 & -2 \end{vmatrix} = (4)(-2) - (-2)(-10) = -8 - 20 = -28$

$$A_y = \begin{bmatrix} 2 & 4 \\ -5 & -10 \end{bmatrix} \Rightarrow \quad |A_y| = \begin{vmatrix} 2 & 4 \\ -5 & -10 \end{vmatrix} = (2)(-10) - (+4)(-5) = -20 + 20 = 0$$

Now $x = \frac{|A_x|}{|A|} = \frac{-28}{-14} = 2 \qquad y = \frac{|A_y|}{|A|} = \frac{0}{-14} = 0$

(viii) $3x - 4y = 4$

$x + 2y = 8$

Solution: (i) The matrix inversion method:

$$3x - 4y = 4$$
$$x + 2y = 8$$

In matrix form: $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ Here, $A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

So, $AX = B \Rightarrow X = A^{-1}\ B$

And $\quad A^{-1} = \frac{1}{|A|}\ \text{Adj}\ A$

$$|A| = \begin{vmatrix} 3 & -4 \\ 1 & 2 \end{vmatrix} = (3)(2) - (1)(-4) = 6 + 4 = 10$$

$$\text{Adj}\ A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$$

So, $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2\times4+4\times8 \\ (-1)\times4+3\times8 \end{bmatrix}$

$$= \frac{1}{10}\begin{bmatrix} 8+32 \\ -4+24 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} \frac{40}{10} \\ \frac{20}{10} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Rightarrow x = 4, y = 2$$

(ii) The cramer's rule:

$$3x - 4y = 4$$
$$x + 2y = 8$$

Here $A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -4 \\ 1 & 2 \end{vmatrix}$$
$$= (2)(3) - (1)(-4) = 6 + 4 = 10$$
$$\Rightarrow |A| = 10$$

Now we find A_x and A_y.

$A_x = \begin{bmatrix} 4 & -4 \\ 8 & 2 \end{bmatrix}$, $A_y = \begin{bmatrix} 4 & 3 \\ 8 & 1 \end{bmatrix}$

$|A_x| = \begin{vmatrix} 4 & -4 \\ 8 & 2 \end{vmatrix}$

$= (2)(4) - (-4)(8) = 8 + 32 = 40$

$|A_y| = \begin{vmatrix} 3 & 4 \\ 1 & 8 \end{vmatrix}$

$= (3)(8) - (1)(4) = 24 - 4 = 20$

$\Rightarrow |A_y| = 20$

Now we find the value of x and y.

$x = \dfrac{|A_x|}{|A|} = \dfrac{40}{10} = 4$

$\Rightarrow x = 4$

$y = \dfrac{|A_y|}{|A|} = \dfrac{20}{10} = 2$

$\Rightarrow y = 2$

$x = 4$, $y = 2$

Solve the following word problems by using

(i) Matrix inversion method (ii) Cramer's rule.

2. The length of a rectangle is 4 times its width.
The perimeter of the rectangle is 150 cm. Find the dimensions of the rectangle.

Solution: If the width of the rectangle is x cm, then length of rectangle is:

$y = 4x$

Or $4x - y = 0$ ______ (1)

Now Perimeter = 150

$2(x + y) = 150$

$x + y = 75$ ______ (2)

(i) In matrix form, we have $\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 75 \end{bmatrix}$

$A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 0 \\ 75 \end{bmatrix}$

$AX = B \Rightarrow X = A^{-1}B$

And $A^{-1} = \dfrac{1}{|A|}\text{Adj } A$

$|A| = \begin{vmatrix} 4 & -1 \\ 1 & 1 \end{vmatrix} = (4)(1) - (1)(-1) = 4 + 1 = 5$

$$\text{Adj } A = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

So ,
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 75 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1\times 0 + 1\times 75 \\ (-1)\times 0 + 4\times 75 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0+75 \\ 0+300 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 75 \\ 300 \end{bmatrix} = \begin{bmatrix} \frac{75}{5} \\ \frac{300}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 60 \end{bmatrix}$$

Hence, length of rectangle = $x = 15$ cm

Width of rectangle = $y = 4x = 4(15) = 60$ cm

(ii) Cramer's Rule:

$4x - y = 0$

$x + y = 75$

Here $A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow \quad |A| = \begin{vmatrix} 4 & -1 \\ 1 & 1 \end{vmatrix} = (4)(1) - (-1)(1) = 4 + 1 = 5$

$A_x = \begin{bmatrix} 0 & -1 \\ 75 & 1 \end{bmatrix} \Rightarrow \quad |A_x| = \begin{vmatrix} 0 & -1 \\ 75 & 1 \end{vmatrix} = (0)(1) - (-1)(75) = 0 + 75 = 75$

$A_y = \begin{bmatrix} 4 & 0 \\ 1 & 75 \end{bmatrix} \Rightarrow \quad |A_y| = \begin{vmatrix} 4 & 0 \\ 1 & 75 \end{vmatrix} = (4)(75) - (0)(1) = 300 - 0 = 300$

Now $x = \frac{|A_x|}{|A|} = \frac{75}{5} = 15 \qquad x = \frac{|A_y|}{|A|} = \frac{300}{5} = 60$

Length of rectangle = 15 cm

Width of rectangle = 60 cm

**3. Two sides of a rectangle differ by 3.5 cm.
Find the dimensions of the rectangle if its perimeter is 67 cm.**

Solution: Let the length of rectangle = x cm.

Let the width of rectangle = y cm.

According to the given condition

$x - y = 3.5$ = ______ (1)

Now Perimeter = 67 $2(x + y) = 67$

$\Rightarrow x + y = 33.5$ ____ (2)

(i) In matrix form, we have $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.5 \\ 33.5 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3.5 \\ 33.5 \end{bmatrix}$

So, $AX = B \Rightarrow X = A^{-1}B$

And $A^{-1} = \frac{1}{|A|}\text{Adj } A$

$|A|=\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=(1)(1)-(-1)(1)=1+1=2$

Now $\text{Adj }A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\Rightarrow A^{-1}=\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$

So, $\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}3.5\\ 33.5\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\times 3.5+1\times 33.5\\ (-1)\times 3.5+1\times 33.5\end{bmatrix}$

$=\frac{1}{2}\begin{bmatrix}3.5+33.5\\ -3.5+33.5\end{bmatrix}=\frac{1}{2}\begin{bmatrix}37\\ 30\end{bmatrix}=\begin{bmatrix}\frac{37}{2}\\ \frac{30}{2}\end{bmatrix}=\begin{bmatrix}18.5\\ 15\end{bmatrix}$

$\Rightarrow$ Length of rectangle $= x = 18.5$ cm
Width of rectangle $= y = 15$ cm

(ii) Cramer's rule:

$x - y = 3.5$
$x + y = 33.5$

Here $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\Rightarrow \quad |A|=\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=(1)(1)-(-1)(1)=1+1=2$

$A_x=\begin{bmatrix}3.5 & -1\\ 33.5 & 1\end{bmatrix}\Rightarrow \quad |A_x|=\begin{vmatrix}3.5 & -1\\ 33.5 & 1\end{vmatrix}=(3.5)(1)-(-1)(33.5)=3.5+33.5=37$

$A_y=\begin{bmatrix}1 & 3.5\\ 1 & 33.5\end{bmatrix}\Rightarrow \quad |A_y|=\begin{vmatrix}1 & 3.5\\ 1 & 33.5\end{vmatrix}=(1)(33.5)-(+1)(3.5)=33.5-3.5=30$

Now $x=\frac{|A_x|}{|A|}=\frac{37}{2}=18.5\text{cm}$

$y=\frac{|A_y|}{|A|}=\frac{30}{2}=15\text{cm}$

4. The third angle of an isosceles triangle is 16° less than the sum of the two equal angles. Find three angles of the triangle.

Solution: Let first and second angles of isosceles triangles are $x = y = x$ and third angle is z.

According to the given condition, we have

$z = (x + x) - 16^\circ$
$z = 2x - 16^\circ$
Or $2x - z = 16^\circ$ ______(1)

We know that

Sum of angles of isosceles triangle $= 180^\circ$

$x + y + z = 180^\circ$
$x + x + z = 180^\circ$
$2x + z = 180^\circ$ _____(2)

(i) In matrix form, we have $\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 16^\circ \\ 180^\circ \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ z \end{bmatrix}, B = \begin{bmatrix} 16^\circ \\ 180^\circ \end{bmatrix}$$

So, $AX = B \Rightarrow X = A^{-1}B$

And $A^{-1} = \frac{1}{|A|}\text{Adj } A$

$$|A| = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = (2)(1) - (2)(-1) = 2 + 2 = 4$$

$$\text{Adj } A = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

So, $\begin{bmatrix} x \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 16^\circ \\ 180^\circ \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1\times 16^\circ + 1\times 180^\circ \\ (-2)\times 16^\circ + 2\times 180^\circ \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 16^\circ + 180^\circ \\ -32^\circ + 360^\circ \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 196^\circ \\ 328^\circ \end{bmatrix}$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \frac{196^\circ}{4} \\ \frac{328^\circ}{4} \end{bmatrix} = \begin{bmatrix} 49^\circ \\ 82^\circ \end{bmatrix}$$

$\Rightarrow x = 49^\circ, y = 49^\circ, z = 82^\circ$

(ii) The cramer's rule:

$2x - z = 16^\circ$
$2x + z = 180^\circ$

Here $A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow \quad |A| = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = (2)(1) - (2)(-1) = 2 + 2 = 4$

$A_x = \begin{bmatrix} 16^\circ & -1 \\ 180^\circ & 1 \end{bmatrix} \Rightarrow \quad |A_x| = \begin{vmatrix} 16^\circ & -1 \\ 180^\circ & 1 \end{vmatrix} = (16^\circ)(1) - (-1)(180^\circ) = 16^\circ + 180^\circ = 196^\circ$

$A_z = \begin{bmatrix} 2 & 16^\circ \\ 2 & 180^\circ \end{bmatrix} \Rightarrow \quad |A_z| = \begin{vmatrix} 2 & 16^\circ \\ 2 & 180^\circ \end{vmatrix} = (2)(180^\circ) - (2)(16^\circ) = 360^\circ - 32^\circ = 328^\circ$

Now $x = \frac{|A_x|}{|A|} = \frac{196^\circ}{4} = 49^\circ \quad z = \frac{|A_z|}{|A|} = \frac{328^\circ}{4} = 82^\circ$

$\Rightarrow \quad x = 49^\circ, \qquad y = 49^\circ, \qquad z = 82^\circ$

5. One acute angle of a right triangle is 12° more than twice the other acute angle. Find the acute angles of the right triangle.

Solution: Let x and y be the two acute angles of a right triangle.

According to the given condition, we have

$x - 12^\circ = 2y$

$x - 2y = +12^\circ$ ______(1)

We know that

Sum of angles of right triangle = 180°

$x + y + z = 180^\circ$

$x + y + 90^\circ = 180^\circ$

$x + y = 180^\circ - 90^\circ = 90^\circ$ ______(2)

(i) In matrix form, we have $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} +12^\circ \\ 90^\circ \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} +12^\circ \\ 90^\circ \end{bmatrix}$$

So, $AX = B \Rightarrow X = A^{-1}B$

And $A^{-1} = \dfrac{1}{|A|}\text{Adj } A$

$$|A| = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = (1)(1) - (1)(-2) = 1 + 2 = 3$$

$$\text{Adj } A = \begin{bmatrix} 1 & +2 \\ -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & +2 \\ -1 & 1 \end{bmatrix}$$

So, $$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} +12^\circ \\ 90^\circ \end{bmatrix} = \frac{1}{3}\begin{bmatrix} (1)\times(12^\circ) + (2)\times(90^\circ) \\ (-1)\times(+12^\circ) + (1)\times(90^\circ) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} +12^\circ + 180^\circ \\ -12^\circ + 90^\circ \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 192^\circ \\ 78^\circ \end{bmatrix} = \begin{bmatrix} \dfrac{192^\circ}{3} \\ \dfrac{78^\circ}{3} \end{bmatrix}$$

$\Rightarrow \quad x = 64^\circ, \qquad y = 26^\circ$

(ii) The cramer's rule:

$x - 2y = +12^\circ$

$x + y = 90^\circ$

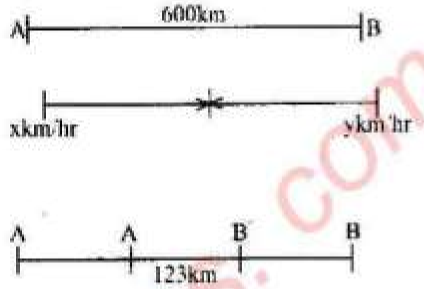
Here $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \Rightarrow \quad |A| = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = (1)(1) - (1)(-2) = 1 + 2 = 3$

$A_x = \begin{bmatrix} +12^\circ & -2 \\ 90^\circ & 1 \end{bmatrix} \Rightarrow |A_x| = \begin{vmatrix} +12^\circ & -2 \\ 90^\circ & 1 \end{vmatrix} = (+12^\circ)(1) - (-2)(90^\circ) = +12^\circ + 180^\circ = 192^\circ$

$A_y = \begin{bmatrix} 1 & +12^\circ \\ 1 & 90^\circ \end{bmatrix} \Rightarrow |A_y| = \begin{vmatrix} 1 & +12^\circ \\ 1 & 90^\circ \end{vmatrix} = (1)(90^\circ) - (1)(+12) = 90^\circ - 12^\circ = 78^\circ$

Now $\quad x = \dfrac{|A_x|}{|A|} = \dfrac{192^\circ}{3} = 64^\circ \qquad y = \dfrac{|A_y|}{|A|} = \dfrac{78^\circ}{3} = 26^\circ$

6. Two cars that are 600 km apart are moving towards each other. Their speeds differs by 6km per hour and the cars are 123 km apart after $4\frac{1}{2}$ hours. Find the speed of each car.



Solution: Let speed of first car = xkm/hr

Speed of second car = ykm/hr

Difference between speed = $x - y = 6$ ------ (i)

Time = $4\frac{1}{2}$ hours = $\frac{9}{2}$ hours

Distance = Speed × Time

$$(x)\left(\frac{9}{2}\right)+123+(y)\left(\frac{9}{2}\right)=600$$

$$\frac{9x}{2}+\frac{9y}{2}=600-123$$

$$\frac{9x}{2}+\frac{9y}{2}=477$$

On multiplying both sides by 2.

$$2\times\left(\frac{9x}{2}+\frac{9y}{2}\right)=477\times 2$$

$$2\times\frac{9x}{2}+2\times\frac{9y}{2}=954$$

$$9x + 9y = 954$$

On dividing both sides by 9.

$$\frac{9x+9y}{9}=\frac{954}{9}$$

$$\frac{9x}{9}+\frac{9y}{9}=106$$

$x + y = 106$ --------- (ii)

(i) The matrix inverse method

$$x - y = 6$$
$$x + y = 106$$

In the matrix form

$$\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}6\\ 106\end{bmatrix}$$ Here, $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}, B=\begin{bmatrix}6\\ 106\end{bmatrix}$

Det of $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=$ $(1)\times(1)-(1)(-1) = 1+1 = 2$

We know that

$$\text{Adj}A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

$X = A^{-1}B$ and $A^{-1} = \dfrac{AdjA}{|A|}$

Hence $\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 106 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1\times 6+1\times 106 \\ -1\times 6+1\times 106 \end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix} 6+106 \\ -6+106 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 112 \\ 100 \end{bmatrix} = \begin{bmatrix} \frac{112}{2} \\ \frac{100}{2} \end{bmatrix} = \begin{bmatrix} 56 \\ 50 \end{bmatrix}$

Thus by the equality of a matrices, speed of first car x =56km/h and the speed of second car y = 50 km/h.

(ii) by using Cramer's rule.

$\Rightarrow \quad x - y = 6$ ---------- (i)

$x + y = 106$ ---------- (ii)

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 106 \end{bmatrix}$

Or AX = B, where

In matrix notation the system can be written in the form

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 106 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1(1)-(1)(-1) = 1+1 = 2 \neq 0$ (non-singular)

Now we find A_x and A_y

$A_x = \begin{bmatrix} 6 & -1 \\ 106 & 1 \end{bmatrix}, \quad A_y = \begin{bmatrix} 1 & 6 \\ 1 & 106 \end{bmatrix}$

$|A_x| = \begin{vmatrix} 6 & -1 \\ 106 & 1 \end{vmatrix}$

$= (1)(6)-(-1)(106) = 6+106 = 112$

$\Rightarrow |A_x| = 112$

$|A_y| = \begin{vmatrix} 1 & 6 \\ 1 & 106 \end{vmatrix}$

$= (1)(106)-(1)(6) = 106-6 = 100$

$|A_y| = 100$

Now we find the value of x and y.

$x = \dfrac{|A_x|}{|A|}$

$x = \dfrac{112}{2} = 56$

$\Rightarrow x = 56$ km/h

$y = \frac{|A_y|}{|A|} = \frac{50}{2} = \frac{100}{2} = 50$

$y = 50$km/h

Hence the speed of each car is 56km/h and 50km/h.

Solved Review Exercise 1

1. Select the correct answer in each of the following.

(i) The order of matrix [2 1] is________

(a) 2–by–1 (b) 1–by–2 (c) 1–by–1 (d) 2–by–2

(ii) $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ is called _______ matrix.

(a) zero (b) unit (c) scalar (d) singular

(iii) Which is order of a square matrix __________

(a) 2–by–2 (b) 1–by–2 (c) 2–by–1 (d) 3–by–2

(iv) Order of transpose of $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$ is __________

(a) 3 – by –2 (b) 2 – by – 3 (c) 1 – by – 3 (d) 3 – by – 1

(v) Adjoint of $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ is __________

(a) $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

(vi) Product of $[x \quad y]\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is __________

(a) $[2x+y]$ (b) $[x-2y]$ (c) $[2x-y]$ (d) $[x+2y]$

(vii) If $\begin{vmatrix} 2 & 6 \\ 3 & x \end{vmatrix} = 0$, then x is equal to ___________

(a) 9 (b) -6 (c) 6 (d) -9

(viii) If $X + \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then X is equal to __________

(a) $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$

Answers: (i) b (ii) c (iii) a (iv) b
(v) a (vi) c (vii) a (viii) d

2. Complete the following:

(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called _______ matrix.

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called _______ matrix.

(iii) Additive inverse of $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ is ______.

(iv) In matrix multiplication, in general, AB ______ BA.

(v) Matrix A + B can be found, if order of A and B is______.

(vi) A matrix is called ______ matrix, if number of rows and columns are equal.

Answers: (i) null (ii) unit (iii) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ (iv) $\neq$ (v) same (vi) square

3. If $\begin{bmatrix} a+3 & 4 \\ 6 & b-1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$, then find 'a' and 'b'.

Solution: $\begin{bmatrix} a+3 & 4 \\ 6 & b-1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$\Rightarrow \quad a+3=-3 \quad$ And $\quad b-1=2$

$a=-3-3 \qquad b=2+1$

$a=-6 \quad , \quad b=3$

4. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$, then find the following.

(i) 2A + 3B

Solution: $2A + 3B = 2\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 15 & -12 \\ -6 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+15 & 6+(-12) \\ 2+(-6) & 0+(-3) \end{bmatrix} = \begin{bmatrix} 19 & -6 \\ -4 & -3 \end{bmatrix}$$

(ii) −3 A + 2B

Solution: $-3A + 2B = -3\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -9 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 10 & -8 \\ -4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -6+10 & -9+(-8) \\ -3+(-4) & 0+(-2) \end{bmatrix} = \begin{bmatrix} 4 & -17 \\ -7 & -2 \end{bmatrix}$$

(iii) − 3 (A + 2B)

Solution: $-3(A+2B) = -3\left(\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}\right) = -3\left(\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 10 & -8 \\ -4 & -2 \end{bmatrix}\right)$

$$= -3\left(\begin{bmatrix} 2+10 & 3-8 \\ 1-4 & 0-2 \end{bmatrix}\right) = -3\begin{bmatrix} 12 & -5 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -36 & 15 \\ 9 & 6 \end{bmatrix}$$

(iv) $\frac{2}{3}(2A - 3B)$

Solution: $\frac{2}{3}(2A-3B) = \frac{2}{3}\left(2\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}\right) = \frac{2}{3}\left(\begin{bmatrix} 4 & 6 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 15 & -12 \\ -6 & -3 \end{bmatrix}\right)$

$$= \frac{2}{3}\left(\begin{bmatrix} 4-15 & 6-(-12) \\ 2-(-6) & 0-(-3) \end{bmatrix}\right) = \frac{2}{3}\begin{bmatrix} -11 & 18 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} -11\times\frac{2}{3} & 18\times\frac{2}{3} \\ 8\times\frac{2}{3} & 3\times\frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{22}{3} & 12 \\ \frac{16}{3} & 2 \end{bmatrix}$$

5. Find the value of X, if $\begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} + X = \begin{bmatrix} 4 & -2 \\ -1 & -2 \end{bmatrix}$

Solution: $\begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} + X = \begin{bmatrix} 4 & -2 \\ -1 & -2 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 4 & -2 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4-2 & -2-1 \\ -1-3 & -2+3 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} -3 & 4 \\ 5 & -2 \end{bmatrix}$, then prove that $AB \neq BA$

Solution: L.H.S. $= AB$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 0\times(-3)+1\times5 & 0\times4+1\times(-2) \\ 2\times(-3)+(-3)\times5 & 2\times4+(-3)\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 0+5 & 0-2 \\ -6-15 & 8+6 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -21 & 14 \end{bmatrix} \text{——(1)}$$

$$BA = \begin{bmatrix} -3 & 4 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} (-3)\times0+4\times2 & (-3)\times1+4\times(-3) \\ 5\times0+(-2)\times2 & 5\times1+(-2)\times(-3) \end{bmatrix} = \begin{bmatrix} 0+8 & -3-12 \\ 0-4 & 5+6 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -15 \\ -4 & 11 \end{bmatrix} \text{——(2)}$$

From (1) and (2), we have
$AB \neq BA$

7. If $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix}$, then verify that

(i) $(AB)^t = B^t A^t$

Solution: L.H.S. $= (AB)^t$

Now $AB = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 3\times2+2\times(-3) & 3\times4+2\times(-5) \\ 1\times2+(-1)\times(-3) & 1\times4+(-1)\times(-5) \end{bmatrix} = \begin{bmatrix} 6-6 & 12-10 \\ 2+3 & 4+5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 5 & 9 \end{bmatrix}$$

So, $(AB)^t = \begin{bmatrix} 0 & 5 \\ 2 & 9 \end{bmatrix}$ ——(1)

R.H.S. $= B^t A^t$

Now, $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix} \Rightarrow B^t = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$

Now $B^tA^t = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2\times3+(-3)\times2 & 2\times1+(-3)\times(-1) \\ 4\times3+(-5)\times2 & 4\times1+(-5)\times(-1) \end{bmatrix}$

$= \begin{bmatrix} 6-6 & 2+3 \\ 12-10 & 4+5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 9 \end{bmatrix}$ ——— (2)

From (1) and (2), we have

$(AB)^t = B^tA^t$

(ii) $\mathbf{(AB)^{-1} = B^{-1}A^{-1}}$

Solution: L.H.S. $= (AB)^{-1}$

Now $AB = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} 3\times2+2\times(-3) & 3\times4+2\times(-5) \\ 1\times2+(-1)\times(-3) & 1\times4+(-1)\times(-5) \end{bmatrix}$

$= \begin{bmatrix} 6-6 & 12-10 \\ 2+3 & 4+5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 5 & 9 \end{bmatrix}$

$|AB| = \begin{vmatrix} 0 & 2 \\ 5 & 9 \end{vmatrix} = (0)(9)-(2)(5) = 0-10 = -10$

$\text{Adj}(AB) = \begin{bmatrix} 9 & -2 \\ -5 & 0 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|}\text{Adj}(AB) = \frac{1}{-10}\begin{bmatrix} 9 & -2 \\ -5 & 0 \end{bmatrix}$ ——— (1)

$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = (3)(-1)-(2)(1) = -3-2 = -5$

$\text{Adj } A = \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix} \Rightarrow |B| = \begin{vmatrix} 2 & 4 \\ -3 & -5 \end{vmatrix} = (2)(-5)-(4)(-3) = -10+12 = 2$

$\text{Adj } B = \begin{bmatrix} -5 & -4 \\ 3 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|}\text{Adj } B = \frac{1}{2}\begin{bmatrix} -5 & -4 \\ 3 & 2 \end{bmatrix}$

So, $B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -5 & -4 \\ 3 & 2 \end{bmatrix}\frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} -5 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$

$= -\frac{1}{10}\begin{bmatrix} (-5)\times(-1)+(-4)\times(-1) & (-5)\times(-2)+(-4)\times3 \\ 3\times(-1)+2\times(-1) & 3\times(-2)+2\times3 \end{bmatrix}$

$$= -\frac{1}{10}\begin{bmatrix} 5+4 & +10-12 \\ -3-2 & -6+6 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} 9 & -2 \\ -5 & 0 \end{bmatrix} \text{——(2)}$$

From (1) and (2), we have

$$(AB)^{-1} = B^{-1}A^{-1}$$

SUMMARY

- A rectangular array of real numbers enclosed within brackets is said to form a matrix.
- A matrix A is called rectangular, if the number of rows and number of columns of A are not equal.
- A matrix A is called a square matrix, if the number of rows of A is equal to the number of columns.
- A matrix A is called a row matrix, if A has only one row.
- A matrix A is called a column matrix, if A has only one column.
- A matrix A is called a null or zero matrix, if each of its entry is 0.
- Let A be a matrix. The matrix A^t is a new matrix which is called transpose of matrix A and is obtained by interchanging rows of A into its respective columns (or columns into respective rows).
- A square matrix A is called symmetric if, $A^t = A$.
- Let A be a matrix. Then its negative, $-A$ is obtained by changing the signs of all the entries of A.
- A square matrix M is said to be skew symmetric, if $M^t = -M$.
- A square matrix M is called a diagonal, matrix if atleast any one of entries of its diagonal is not zero and remaining entries should be zero.
- A diagonal matrix is called identity matrix, if all diagonal entries are 1.
  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called a 3-by-3 identity matrix.
- Any two matrices A and B are called equal if,
  (i) order of A = order of B (ii) corresponding entries are same
- Any two matrices M and N are said to be conformable for addition if order of M = order of N.
- Let A be a matrix of order 2-by-3. Then a matrix B of same order is said to be an additive identity of matrix A, if $B + A = A = A + B$
- Let A be a matrix. A matrix B is defined as an additive inverse of A if $B + A = O = A + B$
- Let A be a matrix. Another matrix B is called the identity matrix of A under multiplication, if $B \times A = A = A \times B$
- Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2-by-2 matrix. A real number λ is called determinant of M,

denoted by det M such that

$$\det M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \lambda$$

- A square matrix M is called singular if the determinant of M is equal to zero.
- A square matrix M is called non-singular if the determinant of M is not equal to zero.
- For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adjoint of M is defined by

  $\text{Adj } M = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- Let M be a square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

  $M^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $\det M = ad - bc \neq 0$.
- The following laws of addition hold as,

  $M + N = N + M$ (Commutative)

  $(M + N) + T = M + (N + T)$ (Associative)
- The matrices M and N are conformable for multiplication to obtain MN if the number of columns of M = number of rows of N, where
  (i) $(MN) \neq NM$, in general
  (ii) $(MN)T = M(NT)$ (Associative law)
  (iii) $M(N+T) = MN + MT$
  (iv) $(N+T)M = NM + TM$ (Distributive laws)
- Law of transpose of product $(AB)^t = B^t A^t$
- $(AB)^{-1} = B^{-1} A^{-1}$
- $AA^{-1} = I = A^{-1}A$
- The solution of a linear system of equations,

  $ax + by = m$

  $cx + dy = n$

  by expressing in the matrix form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$

  is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} m \\ n \end{bmatrix}$

  if the coefficient matrix is non-singular.
- By using the Cramer's rule the determinental form of the solution of equations

  $$x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \text{ where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

UNIT 2

REAL AND COMPLEX NUMBERS

Unit Outlines

STUDENTS LEARNING OUTCOMES

After studying this unit the students will be able to:

- recall the set of real numbers as a union of sets of rational and irrational numbers
- depict real numbers on the number line
- demonstrate a number with terminating and non-terminating recurring decimals on the number line
- give decimal representation of rational and irrational numbers
- know the properties of real numbers
- explain the concept of radicals and radicands
- differentiate between radical form and exponential form of an expression
- transform an expression given in radical form to an exponential form and vice versa.
- recall base, exponent and value
- apply the laws of exponents to simplify expressions with real exponents
- define complex number z represented by an expression of the form $z = a + ib$, where a and b are real numbers and $i = \sqrt{-1}$
- recognize a as real part and b as imaginary part of $z = a + ib$
- define conjugate of a complex number
- know the condition for equality of complex numbers
- carryout basic operations (i.e., addition, subtraction, multiplication and division) on complex numbers

Natural numbers:

The numbers 1, 2, 3,... which we use for counting certain objects are called natural numbers or positive integers. The set of natural numbers is denoted by 'N', i.e., $N = \{1, 2, 3, \ldots\ldots\}$

Whole numbers:

If we include '0' in the set of natural numbers, the resulting set is the set of whole numbers. It is denoted by 'W', i.e., $W = \{0, 1, 2, 3, \ldots\ldots\}$

Integers:

The set of integers consist of positive integers, 0 and negative integers. It is denoted by 'Z'. i.e., $Z = \{\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$

Rational Numbers:

All the numbers of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$ are called rational numbers. It is denoted by 'Q'. i.e., $Q = \left\{\frac{p}{q} \mid p, q \in z \wedge q \neq 0\right\}$.

Irrational Numbers:

The numbers which cannot be expressed as quotient of integers are called irrational numbers. It is denoted by Q'. i.e., $Q' = \left\{x \mid x \neq \frac{p}{q}, p, q \in z \wedge q \neq 0\right\}$.

For example, the numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$ and e are all irrational numbers.

Set of real Numbers:

The union of the set of rational numbers and irrational numbers is known as the set of real numbers. It is denoted by 'R'.i.e., $R = Q \cup Q'$.

Note: Here Q and Q' are both subset of R and $Q \cap Q' = \phi$

Demonstration of a number with terminating and non- terminating decimals on the number line:

(a) Rational Numbers:

The decimal representations of rational numbers are of two types.

(i) Terminating decimal fractions.

(ii) Recurring and non- terminating decimal fractions.

(i) Terminating Decimal Fractions:

The decimal fraction in which there are finite numbers of digits in its decimal part is called a terminating decimal fraction.

For example, $\frac{2}{5} = 0.4$, $\frac{3}{8} = 0.375$, etc.

(ii) Recurring and Non- Terminating Decimal Fractions:

The decimal fraction (non- terminating) in which some digits are repeated again and again in the same order in its decimal part is called a recurring decimal fraction.

For example, $\frac{2}{9} = 0.2222\ldots\ldots$, $\frac{4}{11} = 0.36336\ldots$ etc.

(b) Irrational Numbers:

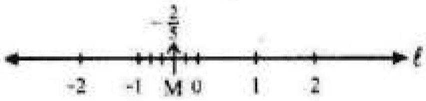
It may be noted that the decimal representations for irrational numbers are neither terminating nor repeating in blocks. The decimal form of an irrational number would continue forever and never begin to repeat the same block of digits.

For example, $\sqrt{2} = 1.414213562\ldots\ldots, \pi = 3.141592654, e = 2.718281829\ldots\ldots$, etc.

Example-1: Present the following numbers on the number line.

(*i*) $-\frac{2}{5}$ (*ii*) $\frac{15}{7}$ (*iii*) $-1\frac{7}{9}$

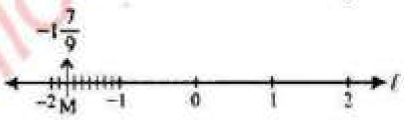
Solution:(i) For representing the rational number '$-\frac{2}{5}$' on the number line l, divide the unit length between o and -1 into five equal parts and take the end of the second part from o to its left side. The point in the following figure represents the rational number '$-\frac{2}{5}$'.



(ii) $\frac{15}{7} = 2 + \frac{1}{7}$. It lies between 2 and 3.

Divide the distance between 2 and 3 into seven equal parts. Te point P represents the point $\frac{15}{7} = 2 + \frac{1}{7}$.

(iii) For representing the rational number $-1\frac{7}{9}$, divide the unit length between -1 and -2 into nine equal parts. Take the end of the 7th part from -1. The point M in the following figure represents the rational number $-1\frac{7}{9}$.



Example-2: Express the following decimals in the form $\frac{p}{q}$ where p, q $\in$ Z and q $\neq$ 0.

(a) $0.\overline{3} = 0.333\cdots$ **(b)** $0.\overline{23} = 0.232323\cdots$

Solution: **(a)** Let $x = 0.\overline{3}$

Which can be written as

$x = 0.3333.....$ ______ (i)

Note that we have only one digit '3' repeating indefinitely.

So, we multiply both sides of eq. (i) by 10, and obtain

$10 \times = (0.3333\ldots) \times 10$

$10 \times = 3.333\ldots$ ______ (ii)

Subtracting (i) from (ii), we have

$10x - x = (3.3333.........) - (0.3333..........)$

$9x = 3.0000$

$x = \frac{3}{9}$

$$x = \frac{1}{3}$$

Hence, $0.\overline{3} = \frac{1}{3}$

(b) Let $x = 0.\overline{23}$

Which can be written as

$$x = 0.23232323 \ldots\ldots \quad \text{\_\_\_\_\_\_\_\_ (i)}$$

Note that we have only two digits block '23' repeating indefinitely.

So, we multiply both sides of (i) by 100, and obtain

$$100\,x = (0.23232323 \ldots\ldots) \times 100$$

$$100x = 23.23232323 \quad \text{\_\_\_\_\_\_\_\_ (ii)}$$

Subtracting (i) from (ii), we have

$$100\,x - x = (23.23232323\ldots\ldots) - (0.23232323 \ldots\ldots)$$

$$99x = 23$$

$$x = \frac{23}{99}$$

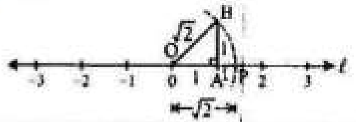
Hence, $0.\overline{23} = \frac{23}{99}$

Decimal Representation of Rational and Irrational Numbers:

Irrational numbers such as $\sqrt{2}, \sqrt{3}$, etc. can be located on the line 'ℓ' by geometric construction. For example, the point corresponding to $\sqrt{2}$ may be constructed by forming a right ΔOAB with sides (containing the right angle) each of length '1' as shown in the figure. By Pythagoras Theorem, we have

$$OB = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

By drawing an arc with centre at O and radius OB $= \sqrt{2}$, we get the point P representing $\sqrt{2}$ on the number line.



Solved Exercise 2.1

1. Identify which of the following are rational and irrational numbers.

(i) $\sqrt{3}$ **(ii)** $\frac{1}{6}$ **(iii)** π **(iv)** $\frac{15}{2}$ **(v)** 7.25 **(vi)** $\sqrt{29}$

Solution:

Rational Number	Irrational Number
(ii) $\frac{1}{6}$	(i) $\sqrt{3}$
(iv) $\frac{15}{2}$	(iii) π
(v) 7.25	(vi) $\sqrt{29}$

2. Convert the following fractions into decimal fractions.

(i) $\frac{17}{25}$

Solution: Let $x = \frac{17}{25}$

Multiply and divide by '4' to make the denominator 100, we get

$$x = \frac{17}{25} \times \frac{4}{4} = \frac{68}{100} = 0.68$$

(ii) $\frac{19}{4}$

Solution: Let $x = \frac{19}{4}$

Multiply and divided by '25' to make the denominator 100, we get

$$x = \frac{19}{4} \times \frac{25}{25} = \frac{475}{100} = 4.75$$

(iii) $\frac{57}{8}$

Solution: Let $x = \frac{57}{8}$

Multiply and divided by '125' to make the denominator 1000, we get

$$x = \frac{57}{8} \times \frac{125}{125} = \frac{7125}{1000} = 7.125$$

(iv) $\frac{205}{18}$

Solution: Let $x = \frac{205}{18}$

Multiply and divided by '55.65' to make the denominator 1000, we get

$$x = \frac{205}{18} \times \frac{55.56}{55.56} = \frac{11389.8}{1000}$$
$$= 11.3898$$

(v) $\frac{5}{8}$

Solution: Let $x = \frac{5}{8}$

Multiply and divided by '125' to make the denominator 1000, we get

$$x = \frac{5}{8} \times \frac{125}{125} = \frac{625}{1000} = 0.625$$

(vi) $\frac{25}{38}$

Solution: Let $x = \frac{25}{38}$

Multiply and divided by '263.1579' to make the denominator 10,000, we get

$$x = \frac{25}{38} \times \frac{263.1579}{263.1579} = \frac{6578.9475}{10,000}$$
$$= 0.65789$$

3. Which of the following statements are true and which are false?

(i) $\frac{2}{3}$ **is an irrational number.** **(ii)** π **is an irrational number.**

(iii) $\frac{1}{9}$ **is a terminating fraction.** **(iv)** $\frac{3}{4}$ **is a terminating fraction.**

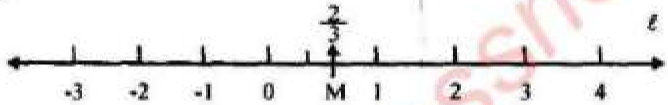
(v) $\frac{4}{5}$ **is a recurring fraction.**

Solution: (i) False (ii) True (iii) False (iv) True (v) False

4. Represent the following numbers on the number line.

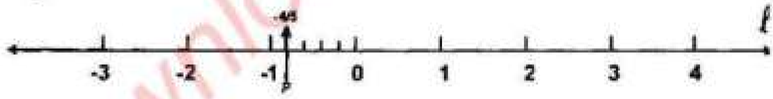
(i) $\frac{2}{3}$

Solution: For representing the rational number $\frac{2}{3}$ on the number line ℓ divide the unit length between 0 and +1 into three equal parts and take the end of the second part from 0 to its right side. The point M in the following figure represents the rational number $\frac{2}{3}$.



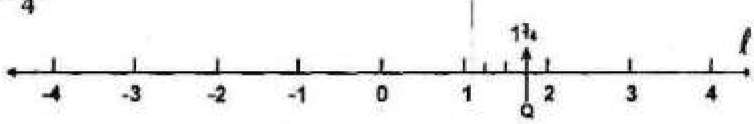
(ii) $-\frac{4}{5}$

Solution: For representing the rational number $-\frac{4}{5}$ on the number line ℓ divide the unit length between 0 and -1 into five equal parts and take the end of the fourth part from 0 to its left side. The point P in following figure represents the rational number $-\frac{4}{5}$.



(iii) $1\frac{3}{4}$

Solution: For representing the rational number $1\frac{3}{4}$ on the number line ℓ divide the unit length between 1 and 2 into four equal parts and take the end of the third part from 1 to its right side. The point Q in following figure represents the rational number $1\frac{3}{4}$.



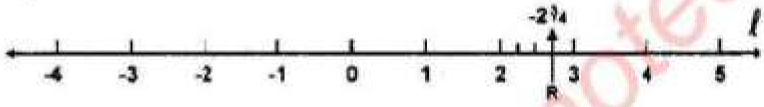
(iv) $-2\frac{5}{8}$

Solution: For representing the rational number $-2\frac{5}{8}$ on the number line ℓ divide the unit length between -2 and -3 into four equal parts and take the end of the fifth part from -2 to its left side. The point M in following figure represents the rational

number $-2\frac{5}{8}$.

(v) $2\frac{3}{4}$

Solution: For representing the rational number $2\frac{3}{4}$ on the number line ℓ divide the unit length between 2 and 3 into four equal parts and take the end of the third part from 2 to its right side. The point R in the following figure represents the rational number $2\frac{3}{4}$.

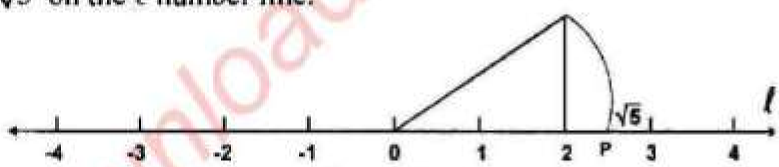


(vi) $\sqrt{5}$

Solution: The point corresponding to $\sqrt{5}$ may be constructed by forming a right ΔOAB with sides (containing the right angle) each of length 1 and 2 as shown in the figure. By Pythagoras Theorem,

$$OB = \sqrt{(1)^2 + (2)^2} = \sqrt{1+4} = \sqrt{5}$$

By drawing an arc with centre at O and radius OB = $\sqrt{5}$, we get the point P representing $\sqrt{5}$ on the ℓ number line.



5. Give a rational number between $\frac{3}{4}$ and $\frac{5}{9}$.

Solution: The mean (average) of any two rational numbers will always be rational and will be between them. So,

$$\frac{\left(\frac{3}{4}+\frac{5}{9}\right)}{2} = \frac{\left(\frac{3\times 9}{4\times 9}+\frac{5\times 4}{9\times 4}\right)}{2} = \frac{\left(\frac{27}{36}+\frac{20}{36}\right)}{2} = \frac{\left(\frac{47}{36}\right)}{2}$$

$$= \frac{\left(\frac{47}{36}\right)}{2} = \frac{47}{36}\times\frac{1}{2} = \frac{47}{72}.$$

6. Express the following recurring decimals as the rational number $\frac{p}{q}$ where p, q are integers and $q \neq 0$.

(i) $0.\overline{5}$

Solution: Let $x = 0.\overline{5}$

Which can be written as

$x = 0.5555 \ldots\ldots$ ________ (i)

Note that we have only one digit 5 repeating indefinitely. So, we multiply both sides of (i) by 10, and obtain

$$10x = (0.5555 \ldots\ldots) \times 10$$
$$10x = 5.000$$
$$x = \frac{5}{9}$$

Hence, $0.\overline{5} = \frac{5}{9}$

(ii) $0.\overline{13}$

Solution: Let $x = (0.\overline{13})$

Which can be written as

$x = (0.13131313 \ldots)$ _____ (i)

Since two digits block 13 is repeating itself indefinitely, so we multiply both sides of (i) by 100, and obtain

$$100x = 0.13131313 \ldots \times 100$$
$100x = 13.1313131313 \ldots$ _____ (ii)

Subtracting (i) from (ii), we have

$$100x - x = (13.1313131313) - (0.1313131313\ldots\ldots)$$
$$99x = 13$$
$$x = \frac{13}{99}$$

Hence, $0.\overline{13} = \frac{13}{99}$

(iii) $0.\overline{67}$

Solution: Let $x = 0.\overline{67}$

Which can be written as

$x = 0.67676767\ldots\ldots\ldots\ldots$ (i)

Since two digits block 67 is repeating itself indefinitely, so we multiply both sides by 100, and obtain

$$100x = (0.67676767\ldots\ldots) \times 100$$
$100x = 67.67676767\ldots\ldots$ _____ (ii)

Subtracting (i) from (ii), we have

$$100x - x = (67.67676767\ldots\ldots) - (0.67676767\ldots\ldots)$$
$$99x = 67$$
$$x = \frac{67}{99}$$

Hence, $0.\overline{67} = \frac{67}{99}$

oooooooooooo

Properties of Real Numbers:

If a, b are real numbers, their sum is written as a + b and their product as ab or a×b a. b or (a) (b).

(a) Properties of Real numbers with respect to Addition and Multiplication:

Properties of real numbers under addition are as follows:

(i) Colosure Property:

$a + b \in R, \forall\ a, b \in R.$

For example, if -3 and $5 \in R$ then $-3 + 5 = 2 \in R$

(ii) Commutative Property:

$a + b = b + a, \ \forall\ a, b \in R.$

For example, if $2, 3 \in R$, then $2 + 3 = 3 + 2 \Rightarrow 5 = 5$

(iii) Associative Property:

$(a+b)+c = a+(b+c), \forall\ a, b, \in R.$

For example, if $5, 7, 3 \in R$, then $(5 + 7) + 3 = 5 + (7 + 3)$

$$12 + 3 = 5 + 10$$
$$15 = 15$$

(iv) Additive Identity:

There exists a unique real number O called additive identity, such that

$a + 0 = a = 0 + a \quad , \forall\ a \in R.$

(v) Additive Inverse:

For every $a \in R$, there exists a unique real number '$-a$' called the additive inverse of a, such that

$a + (-a) = 0 = (-a) + a$

For example, additive inverse of 3 is -3.

Since $3 + (-3) = 0 = (-3) + 3$

Properties of real number under multiplication are as follows:

(i) Closure Property:

$ab \in R \ , \forall\ a, b \in R.$

For example, if $-3, 5 \in R$, then $(-3)(5) \in R \Rightarrow -15 \in R$

(ii) Commutative Property:

$ab = ba, \forall a, b \in R$

For example, if $\frac{1}{3}, \frac{3}{2} \in R$, then $\left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) \Rightarrow \frac{1}{2} = \frac{1}{2}$

(iii) Associative Property:

$(ab)\ c = a(bc), \forall\ a, b, c \in R$

For example, if $2, 3, 5 \in R$, then $(2\times 3)\times 5 = 2\times(3\times 5)$

$$\Rightarrow \quad 6\times 5 = 2\times 15$$
$$30 = 30$$

(iv) Multiplicativative Identity:

There exists a unique real number 1, called the multiplicative identity, such that

$a.1 = a = 1.a, \quad \forall\, a \in R$

(v) Multiplicative Inverse:

For every non-zero real number, there exists a unique real number a^{-1} or $\frac{1}{a}$ is called multiplicative inverse of a.

For example, if $5 \in R$, then $\frac{1}{5} \in R$. Such that

$5 \times \frac{1}{5} = 1 = \frac{1}{5} \times 5$

So, 5 and $\frac{1}{5}$ are multiplicative inverse of each other.

(vi) Multiplication is Distributive Over Addition and Subtraction:

For all a, b, c, $\in R$.

$a(b+c) = ab + ac$ (Left distributive law)

$(a+b)c = ac + bc$ (Right distributive law)

For example; if 2, 3, 5, $\in R$ then

$2(3+5) = 2\times3+2\times5 \Rightarrow 2\times8 = 6+10 \Rightarrow 16 = 16$

And $\forall$ a, b, c $\in R$

$a(b-c) = ab - ac$ (Left distributive law)

$(a-b)c = ac - bc$ (Right distributive law)

For example, if 2, 5, $3 \in R$, then

$2(5-3) = 2\times5 - 2\times3 \Rightarrow 2\times2 = 10-6 \Rightarrow 4 = 4$

Remember: (i) The symbol '$\forall$' mean "for all".

(ii) 'a' is the multiplicative inverse of 'a^{-1}', i.e, $a = (a^{-1})^{-1}$

(b) Properties of Equality of Real Numbers:

Properties of equality of real number are as follows:

(i) Reflexive Property:

$a = a, \quad \forall a \in R.$

(ii) Symmetric Property:

If $a = b$, then $b = a$, $\forall a, b \in R$.

(iii) Transitive Property:

If $a = b$ and $b = c$, then $a = c$, $\forall a, b, c \in R$.

(iv) Additive Property:

If $a = b$, then $a + c = b + c$, $\forall a, b, c \in R$.

(v) Multiplicative Property:

If $a = b$, then $ac = bc$, $\forall a, b, c \in R$.

(vi) Cancellation Property for Addition:

If $a + c = b + c$, then $a = b$, $\forall a, b, c \in R$.

(vii) Cancellation Property for multiplication:

If $ac = bc$, $c \neq 0$ then $a = b$, $\forall a, b, c \in R$.

(c) Properties of Inequalities of Real Numbers:

Properties of inequalities of real numbers are as follows:

(i) Trichotomy Property:

$\forall a, b, \in \mathbb{R}$.

$a < b$ or $a = b$ or $a > b$.

(ii) Transitive Property:

$\forall a, b, c \in \mathbb{R}$.

(a) $a < b$ and $b < c \Rightarrow a < c$ (b) $a > b$ and $b > c \Rightarrow a > c$

(iii) Additive Property:

$\forall\ a, b, c \in \mathbb{R}$.

(a) $a < b \Rightarrow a + c < b + c$

$a < b \Rightarrow c + a < c + b$

(b) $a > b \Rightarrow a + c > b + c$

$a > b \Rightarrow c + a > c + b$

(iv) Multiplicative Property:

(a) $\forall\ a, b, c \in \mathbb{R}$ and $c > 0$.

(i) $a > b \Rightarrow ac > bc$

$a > b \Rightarrow ca > cb$

(ii) $a < b \Rightarrow ac < bc$

$a < b \Rightarrow ca < cb$

(b) $\forall\ a, b, c \in \mathbb{R}$ and $c < 0$

(i) $a > b \Rightarrow ac > bc$

$a > b \Rightarrow ca > cb$

(ii) $a < b \Rightarrow ac < bc$

$a < b \Rightarrow ca < cb$

(v) Multiplicative Inverse:

$\forall\ a, b \in \mathbb{R}$ and $a \neq 0$, $b \neq 0$

(i) $a < b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$ (ii) $a > b \Leftrightarrow \frac{1}{a} < \frac{1}{b}$

Solved Exercise 2.2

1. Identify the property used in the following.

(i) $a + b = b + a$

Solution: Commutative property w.r.t. addition.

(ii) $(ab)c = a(bc)$

Solution: Associative property w.r.t. multiplication.

(iii) $7 \times 1 = 7$

Solution: Multiplicative Identity

(iv) $x > y$ or $x = y$ or $x < y$

Solution: Trichotomy Property.

(v) $ab = ba$

Solution: Commutative property w.r.t. multiplication.

(vi) $a + c = b + c \Rightarrow a = b$

Solution: Cancellation property of addition.

(vii) $5 + (-5) = 0$

Solution: Additive inverse.

(viii) $7 \times \frac{1}{7} = 1$

Solution: Multiplicative inverse.

(ix) $a > b \Rightarrow ac > bc\ (c > 0)$

Solution: Multiplicative property.

2. Fill in the following blanks by stating the properties of real numbers used.

$3x + 3(y - x)$
$= 3x + 3y - 3x,$ ________
$= 3x - 3x + 3y,$ ________
$= 0 + 3y,$ ________
$= 3y$ ________

Solution: $3x + 3(y - x)$

$= 3x + 3y - 3x$ Distributive property
$= 3x - 3x + 3y$ Commutative Property
$= 0 + 3y$ Additive inverse
$= 3y$ Additive identity

3. Give the name of property used in the following.

(i) $\sqrt{24} + 0 = \sqrt{24}$

Solution: Additive Identity

(ii) $-\frac{2}{3}\left(5 + \frac{7}{2}\right) = \left(-\frac{2}{3}\right)(5) + \left(-\frac{2}{3}\right)\left(\frac{7}{2}\right)$

Solution: Distributive Property

(iii) $\pi + (-\pi) = 0$

Solution: Additive Inverse

(iv) $\sqrt{3} \cdot \sqrt{3}$ is a real number

Solution: Closure Property

(v) $\left(-\frac{5}{8}\right)\left(-\frac{8}{5}\right) = 1$

Solution: Multiplicative Inverse.

RADICALS AND RADICANDS

Concept of Radicals and Radicands:

If 'n' is a positive integer greater than '1' and 'a' is a real number, then any real number x, such that $x^n = a$ is called the nth root of a, and in symbols is written as;

$$x = \sqrt[n]{a} \quad \Rightarrow \quad x = a^{\frac{1}{n}}$$

In the radical $\sqrt[n]{a}$, the symbol $\sqrt{\ }$ is called the "radical sign", n is called the "index of the radical" and the real number 'a' under the radical sign is called the "radicand or base".

Remember:

$\sqrt[2]{a}$ is usually written as $\sqrt{a}$.

Difference between Radical form and exponential form:

In radical form, radical sign is used. For example, $x = \sqrt[n]{a}$ is a radical form. $\sqrt[3]{x}, \sqrt[5]{x^2}$ are examples of radical form.

In exponential form, exponential is used in place of radicals. For example, $x = a^{\frac{1}{n}}$ is exponential form.

Properties of radicals:

Let a, b $\in$R and m, n be positive integers,

Then (i) $\sqrt[n]{ab} = \sqrt[n]{a}.\sqrt[n]{b}$ (ii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ (iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

(iv) $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ (v) $\sqrt[n]{a^n} = a$

Transformation of an Expression given in Radical form to Exponential form and vice versa:

Example-1: Write each radical expression in exponential notation and each exponential expression in radical notation.
Do not simplify.

(i) $\sqrt[5]{-8}$ **(ii)** $\sqrt[3]{x^5}$ **(iii)** $y^{3/4}$ **(iv)** $x^{-3/2}$

Solution: (i) $\sqrt[5]{-8} = (-8)^{\frac{1}{5}}$

(ii) $\sqrt[3]{x^5} = \left(x^5\right)^{\frac{1}{3}} = x^{5/3}$

(iii) $y^{\frac{3}{4}} = \left(y^3\right)^{\frac{1}{4}} = \sqrt[4]{y^3}$

(iv) $x^{-\frac{3}{2}} = \left(x^{-3}\right)^{\frac{1}{2}} = \sqrt{x^{-3}}$ or $\left(\sqrt{x}\right)^{-3}$

Example-2: Simplify $\sqrt[3]{16x^4y^5}$

Solution:

$$\begin{aligned}
\sqrt[3]{16x^4y^5} &= \sqrt[3]{(2)(8)(x)\left(x^3\right)\left(y^2\right)\left(y^3\right)} \\
&= \sqrt[3]{2xy^2(2)^3(x)^3(y)^3} \\
&= \sqrt[3]{2xy^2}\sqrt[3]{(2)^3(x)^3(y)^3} \\
&= \sqrt[3]{2xy^2}.\sqrt[3]{2^3}.\sqrt[3]{x^3}.\sqrt[3]{y^3} \\
&= \sqrt[3]{2xy^2}\left(2^3\right)^{\frac{1}{3}}\left(x^3\right)^{\frac{1}{3}}\left(y^3\right)^{\frac{1}{3}} \\
&= \sqrt[3]{2xy^2}(2)(x)(y) \\
&= 2xy\sqrt[3]{2xy^2}
\end{aligned}$$

Solved Exercise 2.3

1. Write each radical expression in exponential notation and each exponential expression in radical notation. Do not simplify.

(i) $\sqrt[3]{-64}$

Solution: $\sqrt[3]{-64} = (-64)^{\frac{1}{3}}$

(ii) $2^{\frac{3}{5}}$

Solution: $2^{\frac{3}{5}} = \left(2^3\right)^{\frac{1}{5}} = \sqrt[5]{2^3}$

(iii) $-7^{\frac{1}{3}}$

Solution: $-7^{\frac{1}{3}} = \sqrt[3]{-7}$

(iv) $y^{-\frac{2}{3}}$

Solution: $y^{-\frac{2}{3}} = \left(y^{-2}\right)^{\frac{1}{3}} = \sqrt[3]{y^{-2}}$

2. Tell whether the following statements are true or false?

(i) $5^{\frac{1}{5}} = \sqrt{5}$ **(ii)** $2^{\frac{2}{3}} = \sqrt[3]{4}$ **(iii)** $\sqrt{49} = \sqrt{7}$ **(iv)** $\sqrt[3]{x^{27}} = x^3$

Solution: (i) False (ii) True (iii) False (iv) False

3. Simplify the following radical expressions.

(i) $\sqrt[3]{-125}$

Solution: $\sqrt[3]{-125} = (-125)^{\frac{1}{3}} = \left(-5^3\right)^{\frac{1}{3}} = -5$

(ii) $\sqrt[4]{32}$

Solution: $\sqrt[4]{32} = (32)^{\frac{1}{4}} = (2\times2\times2\times2\times2)^{\frac{1}{4}}$

$= \left(2\times2^4\right)^{\frac{1}{4}} = 2^{\frac{1}{4}}\times\left(2^4\right)^{\frac{1}{4}} = 2\sqrt[4]{2}$

(iii) $\sqrt[5]{\frac{3}{32}}$

Solution: $\sqrt[5]{\frac{3}{32}} = \frac{\sqrt[5]{3}}{\sqrt[5]{32}} = \frac{\sqrt[5]{3}}{\left(2^5\right)^{\frac{1}{5}}} = \frac{\sqrt[5]{3}}{2}$

(iv) $\sqrt[3]{-\frac{8}{27}}$

Solution: $\sqrt[3]{-\frac{8}{27}} = \left(-\frac{8}{27}\right)^{\frac{1}{3}} = \frac{\left(2^3\right)^{\frac{1}{3}}}{\left(3^3\right)^{\frac{1}{3}}} = -\frac{2}{3}$

LAWS OF EXPONENTS / INDICES

Base and Exponent:

In the exponential notation a^n (read as 'a' to the nth power). We call 'a' as the "base" and 'n' as the "exponent" or the power to which the base is raised.

Laws of Exponents:

If a, b∈R and m, n are positive integers, then

(i) $a^m . a^n = a^{m+n}$ (ii) $\left(a^m\right)^n = a^{mn}$ (iii) $(ab)^n = a^n b^n$

(iv) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$ (v) $\frac{a^m}{a^n} = a^{m-n}$ (vi) $a^0 = 1,\ a \neq 0$

(vii) $a^{-n} = \frac{1}{a^n}, a \neq 0$

Example - 1: Use rules of exponents to simplify each expression and write the answer in terms of positive exponents.

(i) $\frac{x^{-2}x^{-3}y^7}{x^{-3}y^4}$ (ii) $\left(\frac{4a^3b^0}{9a^{-5}}\right)^{-2}$

Solution: (i) $\frac{x^{-2}x^{-3}y^7}{x^{-3}y^4} = \frac{x^{-2-3}y^7}{x^{-3}y^4} = \frac{x^{-5}y^7}{x^{-3}y^4} = \frac{y^7y^{-4}}{x^{-3}.x^5} = \frac{y^{7-4}}{x^{5-3}} = \frac{y^3}{x^2}$

(ii) $\left(\frac{4a^3b^0}{9a^{-5}}\right)^{-2} = \left(\frac{4a^3.a^5.1}{9}\right)^{-2} = \left(\frac{4a^{3+5}}{9}\right)^{-2} = \left(\frac{4a^8}{9}\right)^{-2}$

$= \left(\frac{9}{4a^8}\right)^{+2} = \frac{(9)^2}{(4a^8)^2} = \frac{81}{(4)^2(a^8)^2} = \frac{81}{16a^{16}}$

Example-2: Simplify the following by using laws of indices:

(i) $\left(\frac{8}{125}\right)^{-\frac{4}{3}}$ (ii) $\frac{4(3)^n}{3^{n+1}-3^n}$

Solution: (i) $\left(\frac{8}{125}\right)^{-\frac{4}{3}} = \left(\frac{125}{8}\right)^{\frac{4}{3}} = \frac{(125)^{\frac{4}{3}}}{(8)^{\frac{4}{3}}} = \frac{(5^3)^{\frac{4}{3}}}{(2^3)^{\frac{4}{3}}} = \frac{5^4}{2^4} = \frac{5\times5\times5\times5}{2\times2\times2\times2} = \frac{625}{16}$

(ii) $\frac{4(3)^n}{3^{n+1}-3^n} = \frac{4(3)^n}{3^n(3^1-1)} = \frac{4}{2} = 2$

Solved Exercise 2.4

1. Use laws of exponents to simplify:

(i) $\frac{(243)^{-\frac{2}{3}}(32)^{-\frac{1}{5}}}{\sqrt{(196)^{-1}}}$

Solution: $\frac{(243)^{-\frac{2}{3}}(32)^{-\frac{1}{5}}}{\sqrt{(196)^{-1}}} = \frac{(3^5)^{-\frac{2}{3}}\times(2^5)^{-\frac{1}{5}}}{\sqrt{(2\times2\times7\times7)^{-1}}} = \frac{(3)^{-\frac{10}{3}}\times(2)^{-1}}{(2^2\times7^2)^{-\frac{1}{2}}}$

$= \frac{(2^2\times7^2)^{\frac{1}{2}}}{(3)^{\frac{10}{3}}\times(2)^1} = \frac{(2^2)^{\frac{1}{2}}\times(7^2)^{\frac{1}{2}}}{3^{3+\frac{1}{3}}\times2} = \frac{2\times7}{3^3\times3^{\frac{1}{3}}\times2} = \frac{7}{27\sqrt[3]{3}}$

(ii) $(2x^5y^{-4})(-8x^{-3}y^2)$

Solution: $(2x^5y^{-4})(-8x^{-3}y^2) = (2)(-8)(x^5.x^{-3})(y^{-4}.y^2)$

$$=-16x^{5-3}y^{-4+2}=-16x^2y^{-2}=\frac{-16x^2}{y^2}$$

(iii) $\left(\frac{x^{-2}y^{-1}z^{-4}}{x^4y^{-3}z^0}\right)^{-3}$

Solution: $\left(\frac{x^{-2}y^{-1}z^{-4}}{x^4y^{-3}z^0}\right)^{-3}=\left(x^{-2}.x^{-4}.y^{-1}.y^3.z^{-4}.z^0\right)^{-3}=\left(x^{-2-4}.y^{-1+3}.z^{-4+0}\right)^{-3}$

$$=\left(x^{-6}.y^2.z^{-4}\right)^{-3}=\left(x^{-6}\right)^{-3}\left(y^2\right)^{-3}\left(z^{-4}\right)^{-3}=x^{18}y^{-6}z^{12}=\frac{x^{18}z^{12}}{y^6}$$

(iv) $\frac{(81)^n.3^5-(3)^{4n-1}(243)}{(9^{2n})(3^3)}$

Solution: $\frac{(81)^n.3^5-(3)^{4n-1}(243)}{(9^{2n})(3^3)}=\frac{\left(3^4\right)^n(3)^5-3^{4n-1}(3)^5}{\left(3^2\right)^{2n}\left(3^3\right)}=\frac{3^{4n+5}-3^{4n-1+5}}{3^{4n+3}}$

$$=\frac{3^{4n+5}-3^{4n+4}}{3^{4n+3}}=\frac{3^{4n}\left(3^5-3^4\right)}{3^{4n}.3^3}=\frac{3^5-3^4}{3^3}=\frac{3^3\left(3^2-3^1\right)}{3^3}$$

$$=3^2-3^1=9-3=6$$

2. Show that $\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}=1$

Solution:

L.H.S. $=\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}$

$$=\left(x^a.x^{-b}\right)^{a+b}\times\left(x^b.x^{-c}\right)^{b+c}\times\left(x^c\,.\,x^{-a}\right)^{c+a}$$

$$=\left(x^{a-b}\right)^{a+b}\times\left(x^{b-c}\right)^{b+c}\times\left(x^{c-a}\right)^{c+a}$$

$$=x^{(a-b)(a+b)}.x^{(b-c)(b+c)}.x^{(c-a)(c+a)}=x^{a^2-b^2}.x^{b^2-c^2}.x^{c^2-a^2}$$

$$=x^{a^2-b^2+b^2-c^2+c^2-a^2}$$

$$=1=\text{R.H.S.}$$

Hence $\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}=1$

3. Simplify

(i) $\frac{2^{\frac{1}{3}}\times(27)^{\frac{1}{3}}\times(60)^{\frac{1}{2}}}{(180)^{\frac{1}{2}}\times(4)^{-\frac{1}{3}}\times(9)^{\frac{1}{4}}}$

Solution: $\frac{2^{\frac{1}{3}}\times(27)^{\frac{1}{3}}\times(60)^{\frac{1}{2}}}{(180)^{\frac{1}{2}}\times(4)^{-\frac{1}{3}}\times(9)^{\frac{1}{4}}}=\frac{(2)^{\frac{1}{3}}\times\left(3^3\right)^{\frac{1}{3}}\times\left(2^2\times3\times5\right)^{\frac{1}{2}}}{\left(2^2\times3^2\times5\right)^{\frac{1}{2}}\times\left(2^2\right)^{-\frac{1}{3}}\times\left(3^2\right)^{\frac{1}{4}}}=\frac{2^{\frac{1}{3}}\times3^1\times2^1\times3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{2^1\times3^1\times5^{\frac{1}{2}}\times2^{-\frac{2}{3}}\times3^{\frac{1}{2}}}$

$$=2^{\frac{1}{3}+1-1+\frac{2}{3}}\times3^{1+\frac{1}{2}-1-\frac{1}{2}}\times5^{\frac{1}{2}-\frac{1}{2}}=2^{\frac{1}{3}+\frac{2}{3}}\times3^0\times5^0=2^1\times1\times1=2$$

(ii) $\sqrt{\dfrac{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}{(.04)^{\frac{1}{2}}}}$

Solution: $\sqrt{\dfrac{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}{(.04)^{-\frac{1}{2}}}}=\sqrt{\dfrac{(2^3\times3^3)^{\frac{2}{3}}\times(5^2)^{\frac{1}{2}}}{\left(\frac{4}{100}\right)^{-\frac{1}{2}}}}=\sqrt{\dfrac{(2^3)^{\frac{2}{3}}\times(3^3)^{\frac{2}{3}}\times(5^2)^{\frac{1}{2}}}{\left(\frac{100}{4}\right)^{\frac{1}{2}}}}$

$$=\left(\frac{2^2\times3^2\times5^1}{(25)^{\frac{1}{2}}}\right)^{\frac{1}{2}}=\frac{(2^2)^{\frac{1}{2}}\times(3^2)^{\frac{1}{2}}\times(5^1)^{\frac{1}{2}}}{(5^2)^{\frac{1}{2}\times\frac{1}{2}}}$$

$$=\frac{2\times3\times5^{\frac{1}{2}}}{5^{\frac{1}{2}}}=6$$

(iii) $5^{2^3}\div(5^2)^3$

Solution: $5^{2^3}\div(5^2)^3=5^8\times5^6=5^8\times5^{-6}=5^{8-6}=5^2=25$

(iv) $(x^3)^2\div x^{3^2}, x\neq0$

Solution: $(x^3)^2\div x^{3^2}=x^6+x^9=x^6\times x^{-9}=x^{6-9}=x^{-3}=\dfrac{1}{x^3}$

COMPLEX NUMBERS

Complex number z = a + bi is defined by using imaginary unit $i=\sqrt{-1}$, where a, b$\in$R and a = Re (z), b = Im (z).
For example, number like 3 + 2i, 4 + $3\sqrt{-1}$ are complex numbers.

Integral Powers of i:

By using of $i=\sqrt{-1}\Rightarrow i^2=-1$, we can easily calculate the integral powers of i.
For example, $i^2=-1$, $i^3=i^2\times i=(-1)\times i=-i, i^4=i^2\times i^2=(-1)\times(-1)=1$,

$$i^8=(i^2)^4=(-1)^4=1, i^{10}=(i^2)^5=(-1)^5=-1, \text{etc.}$$

Note: A pure imaginary number is the square root of a negative real number.

Set of complex Numbers:

The set of all complex numbers is denoted by c, and

$$c=\{z\mid z=a+bi, \text{where } a,b\in R \text{ and } i=\sqrt{-1}\}$$

The numbers 'a' and 'b', called the real and imaginary parts of z, are denoted as a= Re (z) and b = Im (z) respectively.

Conjugate of complex Number:

If we change i to $-i$ in z = a + bi, we obtain another complex number a $-$bi called the complex conjugate of z and is denoted by $\bar{z}$ (read as z bar).

The numbers a + b*i* and a – b*i* are called conjugates of each other.

Note that:

(i) $\overline{\overline{z}} = z$

(ii) The conjugate of a real number z = a + 0*i* coincides with the number itself, since. $\overline{z} = \overline{a + 0i} = a - 0i = a$

(iii) Conjugate of a real number is the same real **number.**

Equality of complex Numbers and its Properties:

For all a, b, c, d $\in$R,

$a + bi = c + di$ if and only if a = c and b = d.

For example, $2x + y^2 i = 4 + 9i$

if and only if

$2x = 4$ and $y^2 = 9$

$x = 2$ $\quad y = \pm 3.$

Properties of real numbers R are also valid for the set of complex numbers:

(i) $z_1 = z_1$ (Reflexive law)

(ii) If $z_1 = z_2$, then $z_2 = z_1$ (Symmetric law)

(iii) If $z_1 = z_2$ and $z_2 = z_3$, then $z_1 = z_3$ (Transitive law)

Solved Exercise 2.5

1. Evaluate

(i) i^7

Solution: $i^7 = i^6.i = (i^2)^3.i = (-1)^3 i = (-1)i = -i$

(ii) i^{50}

Solution: $i^{50} = (i^2)^{25} = (-1)^{25} = -1$

(iii) i^{12}

Solution: $i^{12} = (i^2)^6 = (-1)^6 = 1$

(iv) $(-i)^8$

Solution: $(-i)^8 = (-1\times i)^8 = (-1)^8 \times (i)^8 = (1)(i^2)^4 = (1)(-1)^4 = (1)(1) = 1$

(v) $(-i)^5$

Solution: $(-i)^5 = (-1\times i)^5 = (-1)^5 \times (i)^5 = (-1)\times(i^4.i) = (-1)\times(i^2)^2\times(i)$

$= (-1)\times(-1)^2\times i = (-1)\times(1)\times i = -i$

(vi) i^{27}

Solution: $i^{27} = i^{26}.i = (i^2)^{13}.i = (-1)^{13} i = (-1)i = -i$

2. Write the conjugate of the following numbers.

(i) $2 + 3i$

Solution: Let $z = 2 + 3i \Rightarrow \overline{z} = \overline{2+3i} = 2 - 3i$

(ii) $3 - 5i$

Solution: Let $z = 3 - 5i \Rightarrow \overline{z} = \overline{3-5i} = 3 + 5i$

(iii) $-i$

Solution: Let $z=-i \Rightarrow \bar{z}=\overline{-i}=i$

(iv) $-3+4i$

Solution: $z=-3+4i \Rightarrow \bar{z}=\overline{-3+4i}=-3-4i$

(v) $-4-i$

Solution: Let $z=-4-i \Rightarrow \bar{z}=\overline{-4-i}=-4+i$

(vi) $i-3$

Solution: Let $z=i-3 \Rightarrow \bar{z}=\overline{i-3}=-i-3$

3. Write the real and imaginary part of the following numbers.

(i) $1+i$

Solution: Let $z=1+i \Rightarrow \text{Re}(z)=1$ and $\text{Im}(z)=1$

(ii) $-1+2i$

Solution: Let $z=-1+2i \Rightarrow \text{Re}(z)=-1$ and $\text{Im}(z)=2$

(iii) $-3i+2$

Solution: Let $z=-3i+2 \Rightarrow \text{Re}(z)=2$ and $\text{Im}(z)=-3$

(iv) $-2-2i$

Solution: Let $z=-2-2i \Rightarrow \text{Re}(z)=-2$ and $\text{Im}(z)=-2$

(v) $-3i$

Solution: Let $z=0-3i \Rightarrow \text{Re}(z)=0$ and $\text{Im}(z)=-3$

(vi) $2+0i$

Solution: Let $z=2+0i \Rightarrow \text{Re}(z)=2$ and $\text{Im}(z)=0$

4. Find the value of x and y if $x+iy+1=4-3i$.

Solution:
$$x+iy+1=4-3i$$
$$x+iy=4-1-3i$$
$$x+iy=3-3i$$
$$\Rightarrow x=3 \quad \text{and} \quad y=-3$$

BASIC OPERATION ON COMPLEX NUMBERS

(i) Addition:

Let $z_1=a+ib$ and $z_2=c+id$ are two complex numbers and a, b, c, d $\in R$, the sum of two complex numbers is given by

$$z_1+z_2=(a+ib)+(c+id)=a+ib+c+id=a+c+ib+id$$
$$=(a+c)+(b+d)i$$

Therefore, the sum of two complex numbers is the sum of the corresponding real and the imaginary parts. For example,

$$(3-8i)+(5+2i)=3-8i+5+2i$$
$$=3+5-8i+2i=8-6i$$

(ii) Multiplication:

Let $z_1=a+ib$ and $z_2=c+id$ are two complex numbers and a, b, c, d $\in R$, the product of complex number is given by

(i) If $K\in R, kz_1=k(a+bi)=ka+kbi$.

(Multiplication of a complex number with a scalar).

(ii) $z_1z_2 = (a+bi)(c+di) = (ac-bd)+(ad+bc)i$

(Multiplication of two complex numbers)

The multiplication of any two complex numbers $(a + bi)$ and $(c + di)$ is explained as

$$\begin{aligned} z_1z_2 &= (a+bi)(c+di) = a(c+di)+bi(c+di) \\ &= ac+adi+bci+bdi^2 = ac+adi+bci+bd(-1) \\ &= ac+adi+bci-bd = ac-bd+adi+bci \\ &= (ac-bd)+(ad+bc)i \end{aligned}$$

For example, $(2-3i)(4+5i) = 8+10i-12i-15i^2$

$$\begin{aligned} &= 8-2i-15(-1) \\ &= 8-2i+15 = 23-2i \end{aligned}$$

(iii) Subtraction:

Let $z_1 = a+ib$ and $z_2 = c+id$ are two complex numbers and a, b, c, d $\in R$, the difference between two complex numbers is given by

$$z_1 - z_2 = (a+bi)-(c+di) = a+bi-c-di = (a-c)+(b-d)i$$

For example, $(-2+3i)-(2+i) = -2+3i-2-i = -4+2i$

Therefore, the difference of two complex numbers is the difference of the corresponding real and imaginary parts.

(iv) Division:

Let $z_1 = a+ib$ and $z_2 = c+id$ are two complex numbers and a, b, c, d$\in$R, the division of $a + bi$ by $c + di$ is given by

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a+bi}{c+di} = \frac{a+bi}{c+di}\times\frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-(di)^2} = \frac{ac-adi+bci-bd(-1)}{c^2-d^2i^2} \\ &= \frac{ac+bd}{c^2+d^2}+\left(\frac{bc-ad}{c^2+d^2}\right)i \end{aligned}$$

Example - 1: Separate the real and imaginary parts of $\left(-1+\sqrt{-2}\right)^2$.

Solution: Let $z = -1+\sqrt{-2} = -1+i\sqrt{2}$

Then $z^2 = \left(-1+i\sqrt{2}\right)^2 = (-1)^2+2(-1)\left(i\sqrt{2}\right)+\left(i\sqrt{2}\right)^2$

$$= 1-2i\sqrt{2}+i^2\left(\sqrt{2}\right)^2 = 1-2i\sqrt{2}+(-1)(2) = 1-2i\sqrt{2}-2 = -1-2\sqrt{2}i$$

Hence Re $(z^2) = -1$ and Im $(z^2) = -2\sqrt{2}$

Example - 2: Express $\dfrac{1}{1+2i}$ in the standard form $a + bi$.

Solution: $\dfrac{1}{1+2i} = \dfrac{1}{1+2i}\times\dfrac{1-2i}{1-2i} = \dfrac{1-2i}{(1)^2-(2i)^2} = \dfrac{1-2i}{1-4i^2}$

$$=\frac{1-2i}{1-4(-1)}=\frac{1-2i}{1+4}=\frac{1-2i}{5}=\frac{1}{5}-\frac{2}{5}i$$

Example – 3: Express $\frac{4+5i}{4-5i}$ in the standard form a + bi.

Solution: $$\frac{4+5i}{4-5i}=\frac{4+5i}{4-5i}\times\frac{4+5i}{4+5i}=\frac{(4+5i)(4+5i)}{(4-5i)(4+5i)}=\frac{16+20i+20i+25i^2}{(4)^2-(5i)^2}$$

$$=\frac{16+40i-25}{16-25i^2}=\frac{16+40i-25}{16-25(-1)}=\frac{-9+40i}{16+25}=\frac{-9+40i}{41}=-\frac{9}{41}+\frac{40}{41}i$$

Example -4: Solve $(3-4i)(x+yi)=1+0.i$ for real number x and y, where $i=\sqrt{-1}$.
Solution:

$$(3-4i)(x+yi)=1+0i$$
$$3x+3yi-4xi-4yi^2=1+0i$$
$$3x+3yi-4xi-4y(-1)=1+0i$$
$$3x+4y-4xi+3yi=1+0i$$
$$(3x+4y)+(3y-4x)i=1+0i$$

Equating the real and imaginary parts, we obtain

$$3x+4y=1 \quad \text{\_\_\_\_\_\_\_ (1)}$$

And $$3y-4x=0$$

Or $$-4x+3y=0 \quad \text{\_\_\_\_\_\_\_ (2)}$$

4(1) + 3 (2), gives

$$12x+16y=4$$
$$\underline{-12x+9y=0}$$
$$25y=4 \qquad \Rightarrow y=\frac{4}{25}$$

Put 'y' in eq. (1), we get

$$3x+4\left(\frac{4}{25}\right)=1$$
$$3x+\frac{16}{25}=1$$
$$3x=1-\frac{16}{25}$$
$$3x=\frac{9}{25}$$
$$x=\frac{9}{25}\times\frac{1}{3}=\frac{3}{25}$$

Solved Exercise 2.6

1. Identify the following statements as true or false.

(i) $\sqrt{-3}\sqrt{-3} = 3$

(ii) $i^{73} = -i$

(iii) $i^{10} = -1$

(iv) Complex conjugate of $(-6i + i^2)$ is $(-1 + 6i)$.

(v) Difference of a complex number $z = a + bi$ and its conjugate is a real number.

(vi) If $(a-1)-(b+3)i = 5+8i$, then $a = 6$ and $b = -11$

(vii) Product of a complex number and its conjugate is always a non-negative real number.

Solution: (i) False (ii) False (iii) True (iv) True
(v) False (vi) True (vii) True

2. Express each complex number in the standard form a + bi, where a and b are real numbers.

(i) $(2+3i)+(7-2i)$

Solution: $(2+3i)+(7-2i) = 2+3i+7-2i = 2+7+3i-2i = 9+i$

(ii) $2(5+4i)-3(7+4i)$

Solution: $2(5+4i)-3(7+4i) = 10+8i-21-12i = -11-4i$

(iii) $-(-3+5i)-(4+9i)$

Solution: $-(-3+5i)-(4+9i) = 3-5i-4-9i = -1-14i$

(iv) $2i^2+6i^3+3i^{16}-6i^{19}+4i^{25}$

Solution:

$$
\begin{aligned}
2i^2+6i^3+3i^{16}-6i^{9}+4i^{25} &= 2i^2+6i^2.i+3(i^2)^8-6i^{18}.i+4i^{24}.i \\
&= 2i^2+6i^2.i+3(i^2)^8-6(i^2)^9.i+4(i^2)^{12}.i \\
&= 2(-1)+6(-1)i+3(-1)^8-6(-1)^9 i+4(-1)^{12} i \\
&= -2-6i+3(+1)-6(-1)i+4(1)i \\
&= -2-6i+3+6i+4i \\
&= 3-2+6i-6i+4i \\
&= 1+4i
\end{aligned}
$$

3. Simplify and write your answer in the form a + bi.

(i) $(-7+3i)(-3+2i)$.

Solution:

$$
\begin{aligned}
(-7+3i)(-3+2i) &= (-7)(-3)+(-7)(2i)+(3i)(-3)+(3i)(2i) \\
&= 21-14i-9i+6i^2 \\
&= 21-23i+6(-1) \\
&= 21-23i-6 \\
&= 15-23i
\end{aligned}
$$

(ii) $(2-\sqrt{-4})(3-\sqrt{-4})$

Solution:

$$\begin{aligned}\left(2-\sqrt{-4}\right)\left(3-\sqrt{-4}\right) &= (2-2i)(3-2i)\\ &= (2)(3)+(2)(-2i)+(-2i)(3)+(-2i)(-2i)\\ &= 6-4i-6i+4i^2\\ &= 6-10i+4(-1)\\ &= 6-10i-4\\ &= 2-10i\end{aligned}$$

(iii) $\left(\sqrt{5}-3i\right)^2$

Solution:

$$\begin{aligned}\left(\sqrt{5}-3i\right)^2 &= \left(\sqrt{5}-3i\right)\left(\sqrt{5}-3i\right)\\ &= \left(\sqrt{5}\right)\left(\sqrt{5}\right)+\left(\sqrt{5}\right)(-3i)+(-3i)\left(\sqrt{5}\right)+(-3i)(-3i)\\ &= 5-3\sqrt{5}i-3\sqrt{5}i+9i^2\\ &= 5-6\sqrt{5}i+9(-1)\\ &= 5-6\sqrt{5}i-9\\ &= -4-6\sqrt{5}i\end{aligned}$$

(iv) $(2-3i)\left(\overline{3-2i}\right)$

Solution:

$$\begin{aligned}(2-3i)\left(\overline{3-2i}\right) &= (2-3i)(3+2i)\\ &= (2)(3)+(2)(2i)+(-3i)(3)+(-3i)(2i)\\ &= 6+4i-9i-6i^2\\ &= 6-5i-6(-1)\\ &= 6-5i+6\\ &= 12-5i\end{aligned}$$

4. Simplify and write your answer in the form a + b*i*

(i) $\dfrac{-2}{1+i}$

Solution: $$\begin{aligned}\frac{-2}{1+i} &= \frac{-2}{1+i}\times\frac{1-i}{1-i} = \frac{-2(1-i)}{(1+i)(1-i)} = \frac{-2+2i}{(1)^2-(i)^2}\\ &= \frac{-2+2i}{1-(-1)} = \frac{-2+2i}{1+1} = \frac{-2+2i}{2}\\ &= \frac{2(-1+i)}{2} = -1+i\end{aligned}$$

(ii) $\dfrac{2+3i}{4-i}$

Solution:

$$\frac{2+3i}{4-i}=\frac{2+3i}{4-i}\times\frac{4+i}{4+i}=\frac{(2+3i)(4+i)}{(4-i)(4+i)}$$
$$=\frac{(2)(4)+(2)(i)+(3i)(4)+(3i)(i)}{(4)^2-(i)^2}$$
$$=\frac{8+2i+12i+3i^2}{16-(-1)}=\frac{8+14i+3(-1)}{16+1}$$
$$=\frac{8+14i-3}{17}=\frac{5+14i}{17}=\frac{5}{17}+\frac{14}{17}i$$

(iii) $\dfrac{9-7i}{3+i}$

Solution:

$$\frac{9-7i}{3+i}=\frac{9-7i}{3+i}\times\frac{3-i}{3-i}=\frac{(9-7i)(3-i)}{(3+i)(3-i)}$$
$$=\frac{(9)(3)+(9)(i)+(-7i)(3)+(-7i)(-i)}{(3)^2-(i)^2}$$
$$=\frac{27-30i-7}{10}=\frac{20-30i}{10}=\frac{20}{10}-\frac{30}{10}i$$
$$=2-3i$$

(iv) $\dfrac{2-6i}{3+i}-\dfrac{4+i}{3+i}$

Solution:

$$\frac{2-6i}{3+i}-\frac{4+i}{3+i}$$
$$=\frac{(2-6i)-(4+i)}{3+i}=\frac{2-6i-4-i}{3+i}$$
$$=\frac{-2-7i}{3+i}=\frac{-2-7i}{3+i}\times\frac{3-i}{3-i}$$
$$=\frac{(-2-7i)(3-i)}{(3+i)(3-i)}=\frac{(-2)(3)+(-2)(-i)+(-7i)(3)+(-7i)(-i)}{(3)^2-(i)^2}$$
$$=\frac{-6-19i-7}{9+1}=\frac{-13-19i}{10}=\frac{-13}{10}-\frac{19}{10}i$$

(v) $\left(\frac{1+i}{1-i}\right)^2$

Solution: $\left(\frac{1+i}{1-i}\right)^2 = \frac{(1+i)^2}{(1-i)^2} = \frac{(1)^2+2(1)(i)+(i)^2}{(1)^2-2(1)(i)+(i)^2} = \frac{1+2i+i^2}{1-2i+i^2}$

$$= \frac{1+2i+(-1)}{1-2i+(-1)} = \frac{1+2i-1}{1-2i-1} = \frac{2i}{-2i} = -1$$

Alternative Solution:

Now $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i+i+i^2}{(1)^2-(i)^2}$

$$= \frac{1+2i+(-1)}{1-i^2} = \frac{1+2i-1}{1-(-1)} = \frac{2i}{1+1} = \frac{2i}{2} = i$$

So, $\left(\frac{1+i}{1-i}\right)^2 = (i)^2 = i^2 = -1 = -1+0i$

(v) $\frac{1}{(2+3i)(1-i)}$

Solution: $\frac{1}{(2+3i)(1-i)} = \frac{1}{(2)(1)+(3i)(-i)+(2)(-i)+(3i)(1)}$

$$= \frac{1}{2-3i^2-2i+3i} = \frac{1}{2-3(-1)+i} = \frac{1}{2+3+i}$$

$$= \frac{1}{5+i} \times \frac{5-i}{5-i} = \frac{5-i}{(5)^2-(i)^2} = \frac{5-i}{25-i^2}$$

$$= \frac{5-i}{25-(-1)} = \frac{5-i}{25+1} = \frac{5-i}{26} = \frac{5}{26} - \frac{1}{26}i$$

5. Calculate (a) $\bar{z}$ (b) $z+\bar{z}$ (c) $z-\bar{z}$, (d) $z\bar{z}$ for each of the following.

(i) $z = -i$

Solution: (a) $z = -i$

$\bar{z} = \overline{-i} = i$

(b) $z+\bar{z} = -i+i = 0$

(c) $z-\bar{z} = (-i\text{a}) - (i) = -i-i = -2i$

(d) $z\,\bar{z} = (-i)(+i) = -i^2 = +1$

(ii) $z = 2+i$

Solution: $z = 2+i$

(a) $\bar{z} = \overline{2+i} = 2-i$

(b) $z+\bar{z} = (2+i)+(2-i) = 2+i+2-i = 4$

(c) $z-\bar{z}=(2+i)-(2-i)=2+i-2+i=2i$

(d) $z\bar{z}=(2+i)(2-i)=(2)^2-(i)^2=4-i^2=4-(-1)=4+1=5$

(iii) $z=\dfrac{1+i}{1-i}$

Solution: $z=\dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i}=\dfrac{(1+i)(1+i)}{(1-i)(1+i)}=\dfrac{1+i+i+i^2}{1+i-i-i^2}=\dfrac{1+2i-1}{1+1}=\dfrac{2i}{2}=i$

(a) $\bar{z}=\bar{i}=-i$

(b) $z+\bar{z}=i+(-i)=i-i=0$

(c) $z-\bar{z}=i-(-i)=i+i=2i$

(d) $z\bar{z}=(i)(-i)=-i^2=-(-1)=1$

(iv) $z=\dfrac{4-3i}{2+4i}$

Solution: $z=\dfrac{4-3i}{2+4i}=\dfrac{4-3i}{2+4i}\times\dfrac{2-4i}{2-4i}=\dfrac{(4-3i)(2-4i)}{(2+4i)(2-4i)}$

$=\dfrac{8-16i-6i+12i^2}{(2)^2-(4i)^2}=\dfrac{8-22i+12(-1)}{4-16i^2}=-\dfrac{4}{20}-\dfrac{22}{20}i=-\dfrac{1}{5}-\dfrac{11}{10}i$

(a) $\bar{z}=\overline{-\dfrac{1}{5}-\dfrac{11}{10}i}=-\dfrac{1}{5}+\dfrac{11}{10}i$

(b) $z+\bar{z}=\left(-\dfrac{1}{5}-\dfrac{11}{10}i\right)+\left(-\dfrac{1}{5}+\dfrac{11}{10}i\right)=-\dfrac{1}{5}-\dfrac{11}{10}i-\dfrac{1}{5}+\dfrac{11}{10}i=-\dfrac{1}{5}-\dfrac{1}{5}=-\dfrac{2}{5}$

(c) $z-\bar{z}=\left(-\dfrac{1}{5}-\dfrac{11}{10}i\right)-\left(-\dfrac{1}{5}+\dfrac{11}{10}i\right)=-\dfrac{1}{5}-\dfrac{11}{10}i+\dfrac{1}{5}-\dfrac{11}{10}i$

$=\dfrac{-11}{10}i-\dfrac{11}{10}i=\dfrac{-11i-11i}{10}=\dfrac{-22}{10}i=\dfrac{-11}{5}i$

(d) $z\bar{z}=\left(-\dfrac{1}{5}-\dfrac{11}{10}i\right)\times\left(-\dfrac{1}{5}+\dfrac{11}{10}i\right)=\dfrac{1}{25}-\dfrac{11}{50}i+\dfrac{11}{50}i-\dfrac{121}{100}i^2$

$=\dfrac{1}{25}-\dfrac{121}{100}(-1)=\dfrac{1}{25}+\dfrac{121}{100}=\dfrac{4+121}{100}=\dfrac{125}{100}=\dfrac{5}{4}$

6. If $Z=2+3i$ and $W=5-4i$, Show that

(i) $\overline{z+w}=\bar{z}+\bar{w}$

Solution: L. H. S. $=\overline{z+w}$

Now, $z+w=(2+3i)+(5-4i)=2+3i+5-4i=7-i$

So, $\overline{z+w}=\overline{7-i}=7+i$ ______ (1)

R.H.S. $=\bar{z}+\bar{w}$

Now $z = 2 + 3i \Rightarrow \bar{z} = \overline{2+3i} = 2 - 3i$

$w = 5 - 4i \Rightarrow \bar{w} = \overline{5-4i} = 5 + 4i$

So, $\bar{z} + \bar{w} = 2 - 3i + 5 + 4i = 7 + i$ ______ (2)

From (1) and (2), we have

$$\overline{z+w} = \bar{z} + \bar{w}$$

Hence proved

(ii) $\overline{z - w} = \bar{z} - \bar{w}$

Solution: L. H. S. $= \overline{z - w}$

Now, $z - w = (2 + 3i) - (5 - 4i) = 2 + 3i - 5 + 4i = -3 + 7i$

So, $\overline{z-w} = \overline{-3+7i} = -3 - 7i$ ______ (1)

R.H.S. $= \bar{z} - \bar{w}$

Now, $z = 2 + 3i \Rightarrow \bar{z} = \overline{2+3i} = 2 - 3i$

$w = 5 - 4i \Rightarrow \bar{w} = \overline{5-4i} = 5 + 4i$

So, $\bar{z} - \bar{w} = (2 - 3i) - (5 + 4i) = 2 - 3i - 5 - 4i = -3 - 7i$ ______ (2)

From (1) and (2), we have

$$\overline{z-w} = \bar{z} - \bar{w}$$

Hence proved

(iii) $\overline{zw} = \bar{z}\,\bar{w}$

Solution: L. H. S. $= \overline{zw}$

Now, $z\,w = (2 + 3i) \times (5 - 4i) = 10 - 8i + 15i - 12i^2$

$= 10 + 7i - 12(-1) = 10 + 7i + 12 = 22 + 7i$

So, $\overline{zw} = \overline{22+7i} = 22 - 7i$ ______ (1)

R.H.S. $= \bar{z}\,\bar{w}$

Now, $z = 2 + 3i \Rightarrow \bar{z} = \overline{2+3i} = 2 - 3i$

$w = 5 - 4i \Rightarrow \bar{w} = \overline{5-4i} = 5 + 4i$

So, $\bar{z}\,\bar{w} = (2 - 3i) \times (5 + 4i) = 10 + 8i - 15i - 12i^2$

$= 10 - 7i - 12(-1) = 10 - 7i + 12 = 22 - 7i$ ______ (2)

From (1) and (2), we have

$$\overline{zw} = \bar{z}\,\bar{w}$$

Hence proved

(iv) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$, **where $w \neq 0$.**

Solution: L.H.S. $= \overline{\left(\frac{z}{w}\right)}$

$$\text{Now } \frac{z}{w} = \frac{2+3i}{5-4i} = \frac{2+3i}{5-4i} \times \frac{5+4i}{5+4i} = \frac{(2+3i)(5+4i)}{(5-4i)(5+4i)}$$

$$= \frac{10+8i+15i+12i^2}{(5)^2-(4i)^2} = \frac{10+23i+12(-1)}{25-16i^2}$$

$$=\frac{10+23i-12}{25-16(-1)}=\frac{-2+23i}{25+16}=\frac{-2+23i}{41}=\frac{-2}{41}+\frac{23}{41}i$$

So, $\overline{\left(\frac{z}{w}\right)}=\overline{\left(-\frac{2}{41}+\frac{23}{41}i\right)}=-\frac{2}{41}-\frac{23}{41}i$ ______ (1)

R.H.S. $=\frac{\bar{z}}{\bar{w}}$

Now, $z=2+3i \Rightarrow \bar{z}=\overline{2+3i}=2-3i$

$w=5-4i \Rightarrow \bar{w}=\overline{5-4i}=5+4i$

So, $\frac{\bar{z}}{\bar{w}}=\frac{2-3i}{5+4i}=\frac{2-3i}{5+4i}\times\frac{5-4i}{5-4i}=\frac{(2-3i)(5-4i)}{(5+4i)(5-4i)}$

$$=\frac{10-8i-15i+12i^2}{(5)^2-(4i)^2}=\frac{10-23i+12(-1)}{25-16i^2}$$

$$=\frac{10-23i-12}{25-16(-1)}=\frac{-2-23i}{25+16}=\frac{-2-23i}{41}$$

$=-\frac{2}{41}-\frac{23}{41}i$ ______ (2)

From (1) and (2), we have

$$\overline{\left(\frac{z}{w}\right)}=\frac{\bar{z}}{\bar{w}}$$

Hence proved

(v) $\frac{1}{2}(z+\bar{z})$ **is the real part of z.**

Solution: $\frac{1}{2}(z+\bar{z})=\frac{1}{2}((2+3i)+(2-3i))=\frac{1}{2}(2+3i+2-3i)=\frac{1}{2}(4)=2=\text{Re}(z)$

Hence Proved

(vi) $\frac{1}{2i}(z-\bar{z})$ **is the imaginary part of z.**

Solution: $\frac{1}{2i}(z-\bar{z})=\frac{1}{2i}((2+3i)-(2-3i))=(2+3i-2+3i)=\frac{1}{2i}(6i)=3=\text{Im}(z)$

7. Solve the following equations for real x and y.

(i) $(2-3i)(x+yi)=4+i$

Solution: $(2-3i)(x+yi)=4+i$

$2x+2yi-3xi-3yi^2=4+i$

$2x+2yi-3xi-3y(-1)=4+i$

$2x+2yi-3xi+3y=4+i$

$(2x + 3y) + (-3x + 2y)i = 4 + i$

$\Rightarrow \quad 2x + 3y = 4$ ____ (1)

$-3x + 2y = 1$ _____ (2)

By solving eq. (1) and eq. (2), we get

$13y = 14 \Rightarrow \quad y = \dfrac{14}{13}$

Put 'y' in eq. (1), we get

$2x + 3\left(\dfrac{14}{13}\right) = 4$

$2x + \dfrac{42}{13} = 4$

$2x = 4 - \dfrac{42}{13} = \dfrac{52-42}{13}$

$2x = \dfrac{10}{13} \qquad \Rightarrow \qquad x = \dfrac{5}{13}$

(ii) $(3-2i)(x+yi) = 2(x-2yi) + 2i - 1$

Solution: $(3 - 2i)(x + yi) = 2(x - 2yi) + 2i - 1$

$3x + 3yi - 2xi - 2yi^2 = 2x - 4yi + 2i - 1$

$3x + 3yi - 2xi - 2y(-1) = (2x - 1) + (2i - 4yi)$

$(3x + 2y) + (3y - 2x)i = (2x - 1) + (2 - 4y)i$

$\Rightarrow \quad 3x + 2y = 2x - 1 \quad$ and $\quad 3y - 2x = 2 - 4y$

$3x - 2x + 2y = -1 \qquad -2x + 3y + 4y = 2$

$x + 2y = -1$ _____ (1) $\qquad -2x + 7y = 2$ _______ (2)

By solving eq. (1) and eq. (2), we get

$11y = 0 \qquad \Rightarrow \quad y = 0$

Put 'y' in eq. (1), we get

$x + 2(0) = -1$

$x = -1, \; y = 0$

(iii) $(3+4i)^2 - 2(x-yi) = x + yi$

Solution: $(3 + 4i)^2 - 2(x - yi) = x + yi$

$9 + 24i + 16i^2 - 2x + 2yi = x + yi$

$9 + 24i - 16 - 2x + 2yi = x + yi$

$-7 - 2x + 24i + 2yi = x + yi$

$(-7 - 2x) + (24 + 2y)i = x + yi$

$-7 - 2x = x \quad$ and $\quad 24 + 2y = y$

$-2x - x = 7 \qquad 2y - y = -24$

$-3x = 7 \qquad y = -24$

$x = -\dfrac{7}{3}$

Solved Review Exercise 2

1. Multiple Choice Questions. Choose the correct answer:

(i) $\left(27x^{-1}\right)^{-2/3}$ ....

(a) $\frac{\sqrt[3]{x^2}}{9}$ (b) $\frac{\sqrt{x^3}}{9}$ (c) $\frac{\sqrt[3]{x^2}}{8}$ (d) $\frac{\sqrt{x^3}}{8}$

(ii) Write $\sqrt[7]{x}$ in exponential form ....

(a) x (b) x^7 (c) $x^{\frac{1}{7}}$ (d) $x^{\frac{7}{2}}$

(iii) Write $4^{\frac{2}{3}}$ with radical sign ....

(a) $\sqrt[3]{4^2}$ (b) $\sqrt{4^3}$ (c) $\sqrt[2]{4^3}$ (d) $\sqrt{4^6}$

(iv) In $\sqrt[3]{35}$ the radicand is ....

(a) 3 (b) $\frac{1}{3}$ (c) 35 (d) none of these

(v) $\left(\frac{25}{16}\right)^{\frac{-1}{2}}$ = ...

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) $-\frac{5}{4}$ (d) $-\frac{4}{5}$

(vi) The conjugate of $5+4i$ is ....

(a) $-5+4i$ (b) $-5-4i$ (c) $5-4i$ (d) $5+4i$

(vii) The value of i^9 is ....

(a) 1 (b) -1 (c) i (d) $-i$

(viii) Every real number is ....

(a) a positive integer (b) a rational number
(c) a negative integer (d) a complex number

(ix) Real part of $2ab\,(i+i^2)$ is ....

(a) 2ab (b) -2ab (c) $2abi$ (d) $-2abi$

(x) Imaginary part of $-i\,(3i+2)$ is ....

(a) -2 (b) 2 (c) 3 (d) -3

(xi) Which of the following sets have the closure property w.r.t. addition ....

(a) $\{0\}$ (b) $\{0,-1\}$ (c) $\{0,1\}$ (d) $\left\{1,\sqrt{2},\frac{1}{2}\right\}$

(xii) Name the property of real numbers used in $\left(-\frac{\sqrt{5}}{2}\right)\times 1=-\frac{\sqrt{5}}{2}$ ....

(a) additive identity (b) additive inverse
(c) multiplicative identity (d) multiplicative inverse

(xiii) If $x, y, z \in R$, $z<0$ then, $x<y \Rightarrow$ ....

(a) $xz<yz$ (b) $xz>yz$ (c) $xz=yz$ (d) none of these

(xiv) If $a, b \in R$, then only one of $a=b$ or $a<b$ or $a>b$ holds is called ....

(a) trichotomy property (b) transitive property

(c) additive property (d) multiplicative property

(xv) A non- terminating, non- recurring decimal represents.
(a) a natural number (b) a rational numbers
(c) an irrational number (d) a prime number

Solution:

(i)	a	**(ii)**	c	**(iii)**	a	**(iv)**	c	**(v)**	b	**(vi)**	c
(vii)	c	**(viii)**	d	**(ix)**	b	**(x)**	a	**(xi)**	a	**(xii)**	c
(xiii)	b	**(xiv)**	a	**(xv)**	c						

2. True or false? Identify.

(i) Division is not an associative operation. ______
(ii) Every whole number is a natural number. ______
(iii) Multiplicative inverse of 0.02 is 50. ______
(iv) π is a rational number. ______
(v) Every integer is a rational number. ______
(vi) Subtraction is a commutative operation. ______
(vii) Every real number is a rational number. ______
(viii) Decimal representation of a rational number is either terminating or recurring. ______
(ix) $1.\overline{8} = 1 + \frac{8}{9}$ ______

Answers:

(i)	True	(ii)	False	(iii)	True	(iv)	False	(v)	True
(vi)	False	(vii)	False	(viii)	True	(ix)	True		

Simplify the following.

(i) $\sqrt[4]{81y^{12}x^{8}}$

Solution: $\sqrt[4]{81y^{-12}x^{-8}} = \left(81y^{-12}x^{-8}\right)^{\frac{1}{4}} = \left(3^4y^{-12}x^{-8}\right)^{\frac{1}{4}}$

$$= \left(3^4\right)^{\frac{1}{4}}\left(y^{-12}\right)^{\frac{1}{4}}\left(x^{-8}\right)^{\frac{1}{4}} = 3y^{-3}x^{-2} = \frac{3}{x^2y^3}$$

(ii) $\sqrt{25x^{10n}y^{8m}}$

Solution: $\sqrt{25x^{10n}y^{8m}} = \left(25x^{10n}y^{8m}\right)^{\frac{1}{2}} = \left(5^2x^{10n}y^{8m}\right)^{\frac{1}{2}}$

$$= \left(5^2\right)^{\frac{1}{2}}\left(x^{10n}\right)^{\frac{1}{2}}\left(y^{8m}\right)^{\frac{1}{2}} = 5x^{5n}y^{4m}$$

(iii) $\left(\frac{x^3.y^4.z^5}{x^2.y^1.z^5}\right)^{\frac{1}{5}}$

Solution: $\left(\frac{x^3.y^4.z^5}{x^{-2}.y^{-1}.z^{-5}}\right)^{\frac{1}{5}} = \left(x^3.x^2.y^4.y^1.z^5.z^5\right)^{\frac{1}{5}}$

$$= \left(x^{3+2}.y^{4+1}.z^{5+5}\right)^{\frac{1}{5}} = \left(x^5.y^5.z^{10}\right)^{\frac{1}{5}} = \left(x^5\right)^{\frac{1}{5}}\left(y^5\right)^{\frac{1}{5}}\left(z^{10}\right)^{\frac{1}{5}}$$
$$= xyz^2$$

(iv) $\left(\dfrac{32x^{-6}y^{-4}z}{625x^4yz^{-4}}\right)^{\frac{2}{5}}$

Solution: $\left(\dfrac{32x^{-6}y^{-4}z}{625x^4yz^{-4}}\right)^{\frac{2}{5}} = \left(\left(\dfrac{2^5}{5^4}\right)x^{-6}.x^{-4}.y^{-4}.y^{-1}.z^1.z^4\right)^{\frac{2}{5}}$

$= \left(\dfrac{2^5}{5^4}.x^{-6-4}y^{-4-1}z^{1+4}\right)^{\frac{2}{5}} = \left(\dfrac{2^5}{5^4}x^{-10}y^{-5}z^5\right)^{\frac{2}{5}}$

$= \dfrac{\left(2^5\right)^{\frac{2}{5}}}{\left(5^4\right)^{\frac{2}{5}}}\left(x^{-10}\right)^{\frac{2}{5}}\left(y^{-5}\right)^{\frac{2}{5}}\left(z^5\right)^{\frac{2}{5}} = \dfrac{2^2}{5^{\frac{8}{5}}}(x)^{-4}(y)^{-2}(z)^2 = \dfrac{4z^2}{5\times5^{\frac{3}{5}}x^4y^2}$

4. Simplify $\sqrt{\dfrac{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}{(0.04)^{-3/2}}}$

Solution:

$$\sqrt{\frac{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}{(0.04)^{-3/2}}} = \sqrt{\frac{\left(2^3\times3^3\right)^{\frac{2}{3}}\times\left(5^2\right)^{\frac{1}{2}}}{\left(\frac{4}{100}\right)^{-\frac{3}{2}}}} = \sqrt{\frac{\left(2^3\right)^{\frac{2}{3}}\times\left(3^3\right)^{\frac{2}{3}}\times\left(5^2\right)^{\frac{1}{2}}}{\left(\frac{1}{25}\right)^{-\frac{3}{2}}}}$$

$$= \left(\frac{2^2\times3^2\times5^1}{(25)^{\frac{3}{2}}}\right)^{\frac{1}{2}} = \left(\frac{2^2\times3^2\times5^1}{\left(5^2\right)^{\frac{3}{2}}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{2^2\times3^2\times5^1}{5^3}\right)^{\frac{1}{2}} = \left(\frac{2^2\times3^2}{5^{3-1}}\right)^{\frac{1}{2}} = \left(\frac{2^2\times3^2}{5^2}\right)^{\frac{1}{2}}$$

$$= \frac{\left(2^2\right)^{\frac{1}{2}}\times\left(3^2\right)^{\frac{1}{2}}}{\left(5^2\right)^{\frac{1}{2}}} = \frac{2\times3}{5} = \frac{6}{5}$$

5. Simplify $\left(\dfrac{a^p}{a^q}\right)^{p+q}.\left(\dfrac{a^q}{a^r}\right)^{q+r}\div5\left(a^p.a^r\right)^{p-r}$, $a\neq0$

Solution: $\left(\dfrac{a^p}{a^q}\right)^{p+q}.\left(\dfrac{a^q}{a^r}\right)^{q+r}\div5\left(a^p.a^r\right)^{p-r}$

$= \left(a^p.a^{-q}\right)^{p+q}.\left(a^q.a^{-r}\right)^{q+r}\div5\left(a^{p+r}\right)^{p-r} = \left(a^{p-q}\right)^{p+q}.\left(a^{q-r}\right)^{q+r}\div5a^{p^2-r^2}$

$= a^{p^2-q^2}.a^{q^2-r^2}\times\dfrac{1}{5}a^{-p^2+r^2} = \dfrac{1}{5}a^0 = \dfrac{1}{5}(1) = \dfrac{1}{5}$

6. **Simplify** $\left(\dfrac{a^{2\ell}}{a^{\ell+m}}\right)\left(\dfrac{a^{2m}}{a^{m+n}}\right)\left(\dfrac{a^{2n}}{a^{n+\ell}}\right)$

Solution:

$$\left(\frac{a^{2\ell}}{a^{\ell+m}}\right)\left(\frac{a^{2m}}{a^{m+n}}\right)\left(\frac{a^{2n}}{a^{n+\ell}}\right)=\left(a^{2\ell}.a^{-\ell-m}\right)\left(a^{2m}.a^{-m-n}\right)\left(a^{2n}.a^{-n-\ell}\right)$$
$$=\left(a^{2\ell-\ell-m}\right)\left(a^{2m-m-n}\right)\left(a^{2n-n\ell}\right)$$
$$=\left(a^{\ell-m}\right)\left(a^{m-n}\right)\left(a^{n-\ell}\right)$$
$$=a^{\ell-m+m-n+n-\ell}$$
$$=a^{o}=1$$

7. **Simplify** $\sqrt[3]{\dfrac{a^{\ell}}{a^{m}}}\times\sqrt[3]{\dfrac{a^{m}}{a^{n}}}\times\sqrt[3]{\dfrac{a^{n}}{a^{\ell}}}$

Solution:

$$\sqrt[3]{\frac{a^{\ell}}{a^{m}}}\times\sqrt[3]{\frac{a^{m}}{a^{n}}}\times\sqrt[3]{\frac{a^{n}}{a^{\ell}}}=\left(\frac{a^{\ell}}{a^{m}}\right)^{\frac{1}{3}}\times\left(\frac{a^{m}}{a^{n}}\right)^{\frac{1}{3}}\times\left(\frac{a^{n}}{a^{\ell}}\right)^{\frac{1}{3}}$$
$$=\left(a^{\ell}.a^{-m}\right)^{\frac{1}{3}}\times\left(a^{m}.a^{-n}\right)^{\frac{1}{3}}\times\left(a^{n}.a^{-\ell}\right)^{\frac{1}{3}}$$
$$=\left(a^{\ell}.a^{-m}\right)^{\frac{1}{3}}\times\left(a^{m-n}\right)^{\frac{1}{3}}\times\left(a^{n-l}\right)^{\frac{1}{3}}=a^{\frac{1}{3}(\ell+m)}\times a^{\frac{1}{3}(m-n)}\times a^{\frac{1}{3}(n-\ell)}$$
$$=a^{\frac{1}{3}(l-m)+\frac{1}{3}(m-n)+\frac{1}{3}(n-\ell)}=a^{\frac{1}{3}(l-m+m-n+n-\ell)}=a^{\frac{1}{3}(0)}=a^{o}=1$$

SUMMARY

- Set of real numbers is expressed as $R = Q \cup Q'$, where
  $Q=\left\{\dfrac{p}{q}\,\middle|\,p,q\in Z, q\neq 0\right\}, Q'=\{x\,|\,x \text{ is not rational}\}$
- Properties of real numbers w.r.t. addition and multiplication are as.
  Closure: $a + b \in R$, $ab \in R$, $\forall$ a, b $\in$ R
  Associative: $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$, $\forall$ a, b, c $\in$ R
  Commutative: $a + b = b + a$, $ab = ba$, $\forall$ a, b, c $\in$ R
  Additive Identity: $a + 0 = a = 0 + a$, $\forall$ a $\in$ R,
  Multiplicative Identity: $a.1 = a = 1 . a$, $\forall$a $\in$ R,
  Additive Inverse: $a + (-a) = 0 = (-a) + a$, $\forall$ a $\in$ R,
  Multiplicative Inverse: $a.\dfrac{1}{a}=1=\dfrac{1}{a}.a$, $a\neq 0$
  Multiplication is distributive over addition and subtraction:
  $a(b + c) = ab + ac$, $\forall$ a, b, c $\in$ R
  $(b + c)a = ba + ca$, $\forall$ a, b, c $\in$ R

$a(b-c) = ab - ac,\ \forall\ a, b, c \in R$
$(a-b)c = ac - bc,\ \forall\ a, b, c \in R$

- Properties of equality in R
  Reflexive: $a = a,\ \forall\ a \in R$
  Symmetric $a = b \Rightarrow b = a,\ \forall\ a, b \in R$
  Transitive $a = b, b = c \Rightarrow a = c\ \forall\ a, b, c \in R$
  Additive property: If $a = b$, then $a + c = b + c,\ \forall\ a, b, c \in R$
  Multiplicative property: If $a = b$, then $ac = bc;\ \forall\ a, b, c \in R$
  Concellation property: If $ac = bc$, $c \neq 0$, then $a = b,\ \forall\ a, b, c \in R$
- In the radical $\sqrt[n]{x}, \sqrt{\ }$ is radical sign, x is radicand or base and n is index of radical.
- Indices and laws of indices:
  $\forall\ a, b, c \in R$ and $m, n \in Z$

  $(a^m)^n = a^{mn}, (ab)^n = a^n b^n$

  $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$

  $a^m a^n = a^{m+n}$

  $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$

  $a^{-n} = \frac{1}{a^n}, a \neq 0$

  $a^0 = 1$
- Complex number $z = a + bi$ is defined using imaginary unit $i = \sqrt{-1}$, where
  $a, b \in R$ and $a = \text{Re}(z)$, $b = \text{Im}(z)$
- Conjugate of $z = a + bi$ is defined as $\overline{Z} = a - bi$

UNIT 3

LOGARITHMS

Unit Outlines			
3.1	*Scientific Notation*	3.2	*Logarithm*
3.3	*Common and Naturlal Logarithm*	3.4	*Laws of Logarithm*
3.5	*Application of Logarithm*		

STUDENTS LEARNING OUTCOMES

After studying this unit the students will be able to:

- express a number in standard form of scientific notation and vice versa.
- define logarithm of number " y "to the base 'a' as the power to which a must be raised to give the number (i.e., $a^x = y \Leftrightarrow \log_a y = x$, $a > 0$, $a \neq 1$ and $y > 0$).
- define a common logarithm, characteristic and mantissa of log of a number.
- use tables to find the log of a number.
- give concept of antilog and use tables to find the antilog of a number.
- differentiate between common and natural logarithm.
- prove the following laws of logarithm.
  - ★ $\log_a(mn) = \log_a m + \log_a n$
  - ★ $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
  - ★ $\log_a m^n = n\log_a m$
  - ★ $\log_a m \log_m n = \log_a n$
- apply laws of logarithm to convert lengthy processes, of multiplication, division and exponentiation into easier processes of addition and subtraction etc.

Introduction:

The difficult and complicated calculations become easier by using logarithms. Abu Muhammad Musa Al Khwarizmi first gave the idea of logarithms. Later on, in the seventeenth century John Napier extended his work on logarithms and prepared tables for logarithms. He used "*e*" as the base for the preparation of logarithm tables. Professor Henry Briggs had a special interest in the work of John Napier. He prepared logarithm tables with base 10. Antilogarithm table was prepared by Jobst Burgi in 1620 A.D.

Scientific Notation:

There are so many numbers that we use in science and technical work that are either very small or very large. For instance, the distance form the Earth to the Sun is 150,000,000 km approx. and a hydrogen atom weighs 0.000,000,000,000,000.000.000. 001. 7 gram. While writing these numbers in ordinary notation (standard notation) there is always chance of making an error by omitting a zero or writing more than actual

number of zeros. To overcome this problem, scientists have developed a concise and convenient method to write very small or very large numbers, that is called scientific notation of expressing an ordinary number.

A number written in the form $a \times 10^n$, where $1 \leq a < 10$ and 'n' is an integer, is called the scientific notation.

The above mentioned number can be conveniently written in scientific notation as 1.5×10^8 km and 1.7×10^{-24} gm respectively.

Example-1: Write each of the following ordinary numbers in scientific notation.

(i) 30600 (ii) 0.000058

Solution: (i) $30600 = 3.06 \times 10^4$ (move decimal point four places to the left)

(ii) $0.000058 = 5.8 \times 10^{-5}$ (move decimal point five places to the right)

Observe that for expressing a number in scientific notation:

(i) Place the decimal point after the first non-zero digit of given number.

(ii) We multiply the number obtained in step (i), by 10^n if we shifted the decimal point n places to the left.

(iii) We multiply the number obtained in step (i) by 10^{-n} if we shifted the decimal point n places to the right.

On the other hand, if we want to change a number from scientific notation to ordinary (standard) notation we simply reverse the process.

Example-2: Change each of the following numbers from scientific notation to ordinary notation. (i) 6.35×10^6 (ii) 7.61×10^{-4}

Solution: (i) $6.35 \times 10^6 = 6350000$ (move the decimal point six places to the right)

(ii) $7.61 \times 10^{-4} = 0.000761$ (move the decimal point six places to the left)

Solved Exercise 3.1

1. Express each of the following numbers in scientific notation.

(i) 5700

Solution: $5700 = 57 \times 100 = 5.7 \times 10^1 \times 10^2 = 5.7 \times 10^{1+2} = 5.7 \times 10^3$

(ii) 49,800,000

Solution: $49,800,000 = 498 \times 100000 = 4.98 \times 10^2 \times 10^5 = 4.98 \times 10^{2+5} = 4.98 \times 10^7$

(iii) 96, 000, 000

Solution: $96,000,000 = 96 \times 1000000 = 9.6 \times 10^1 \times 10^6 = 9.6 \times 10^{1+6} = 9.6 \times 10^7$

(iv) 416.9

Solution: $416.9 = \frac{4169}{10} = 4169 \times 10^{-1} = 4.169 \times 10^3 \times 10^{-1} = 4.169 \times 10^{3-1} = 4.169 \times 10^2$

(v) 83,000

Solution: $83,000 = 83 \times 1000 = 8.3 \times 10^1 \times 10^3 = 8.3 \times 10^{1+3} = 8.3 \times 10^4$

(vi) 0.00643

Solution: $0.00643 = \frac{643}{100000} = 643 \times 10^{-5} = 6.43 \times 10^2 \times 10^{-5} = 6.43 \times 10^{2-5} = 6.43 \times 10^{-3}$

(vii) 0.0074

Solution: $0.0074 = \frac{74}{10000} = 74 \times 10^{-4} = 7.4 \times 10^{1} \times 10^{-4} = 7.4 \times 10^{1-4} = 7.4 \times 10^{-3}$

(viii) 60,000,000

Solution: $60{,}000{,}000 = 6 \times 10{,}000{,}000 = 6 \times 10^{7}$

(ix) 0.0000000395

Solution: $0.0000000395 = \frac{395}{100000000000} = 395 \times 10^{-11} = 3.95 \times 10^{2} \times 10^{-11}$

$= 3.95 \times 10^{2-11} = 3.95 \times 10^{-9}$

(x) $\frac{275{,}000}{0.0025}$

Solution: $\frac{275{,}000}{0.0025} = \frac{275 \times 1000}{\frac{25}{10000}} = \frac{2.75 \times 10^{2} \times 10^{3}}{25 \times 10^{-4}} = \frac{2.75 \times 10^{2+3}}{2.5 \times 10^{1} \times 10^{-4}}$

$= \frac{2.75 \times 10^{5}}{2.5 \times 10^{1-4}} = \frac{2.75 \times 10^{5}}{2.5 \times 10^{-3}}$

2. Express the following numbers in ordinary notation.

(i) 6×10^{-4}

Solution: $6 \times 10^{-4} = \frac{6}{10^{4}} = \frac{6}{10000} = 0.0006$

(ii) 5.06×10^{10}

Solution: $5.06 \times 10^{10} = \frac{506}{100} \times 10000000000 = 506 \times 100000000 = 50{,}600{,}000{,}000$

(iii) 9.018×10^{-6}

Solution: $9.018 \times 10^{-6} = \frac{9.018}{10^{6}} = \frac{9018}{1000 \times 1000000} = \frac{9018}{1000000000} = 0.000009018$

(iv) 7.865×10^{8}

Solution: $7.865 \times 10^{8} = \frac{7865}{1000} \times 10^{8} = \frac{7865}{10^{3}} \times 10^{8} = 7865 \times 10^{8} \times 10^{-3}$

$= 7865 \times 10^{8-3} = 7865 \times 10^{5} = 7865 \times 100000 = 786{,}500{,}000$

LOGARITHM

Logarithms are useful tools for accurate and rapid computations. Logarithms with base 10 are known as common logarithms and those with base e are known as natural logarithms. We shall define logarithms with base $a > 0$ and $a \neq 1$.

Logarithm of a Real Number:

If $a^{x} = y$ then x is called the logarithm of y to the base 'a' and is written as $\log_a y = x$, where $a > 0$, $y > 0$ and $a \neq 1$. i.e., the logarithm of a number y to the base 'a' is the index x of the power to which the base must be raised to get that number y.

The relation $a^{x} = y$ and $\log_a y = x$ are equivalent. When one relation is given, it can be converted into the other. Thus

$$\boxed{a^{x} = y \Leftrightarrow \log_a y = x}$$

The form $a^x = y$ on the left is called the exponential form, whereas the form $\log_a y = x$ on the right hand side is called the logarithmic form of the same relation.

To explain these remarks, we observe that
$3^2 = 9$ is equivalent to $\log_3 9 = 2$ and $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$.
Similarly, we can say that

$\text{Log}_3 27 = 3$ is equivalent to $27 = 3^3$

Remember: Logarithm of a negative number is not defined since the base is always taken to be positive.

Example-3: Find $\log_4 2$, i.e., find log of 2 to the base 4.

Solution: Let $\log_4 2 = x$ $\Rightarrow$ $4^x = 2$

i.e., $2^{2x} = 2^1$ $\Rightarrow$ $2x = 1$

or $x = \frac{1}{2}$ $\Rightarrow$ $\log_4 2 = \frac{1}{2}$

Deductions from Definitions of Logarithm:

(1) Since $a^0 = 1$, $\log_a 1 = 0$ (2) Since $a^1 = a$, $\log_a a = 1$

Definitions of Common Logarithms, Characteristic and Mantissa

(a) Common Logarithms:

In numerical calculations, the base of logarithm is always taken as 10. These logarithms are called common logarithms or Briggesian logarithms in honour of Henry Briggs, an English mathematician and astronomer, who developed them.

(b) Characteristic and Mantissa of Log of a Number:

Consider the following:

$10^3 = 1000 \Leftrightarrow \log 1000 = 3$
$10^2 = 100 \Leftrightarrow \log 100 = 2$
$10^1 = 10 \Leftrightarrow \log 10 = 1$
$10^0 = 1 \Leftrightarrow \log 1 = 0$
$10^{-1} = 0.1 \Leftrightarrow \log 0.1 = -1$
$10^{-2} = 0.01 \Leftrightarrow \log 0.01 = -2$
$10^{-3} = 0.001 \Leftrightarrow \log 0.001 = -3$

Remember: By convention, if only the common logarithms are used throughout a discussion, the base 10 is not written.

Also consider the following table.

For the numbers	The logarithm is
Between 1 and 10	a decimal
Between 10 and 100	1 + a decimal
Between 100 and 1000	2 + a decimal
Between 0.1 and 1	−1 + a decimal
Between 0.01 and 0.1	−2 + a decimal
Between 0.001 and 0.01	−3 + a decimal

Observe that:

The logarithm of any number consists of two parts:

(i) an integral part which is positive for a number greater than 1 and negative for a number less than 1 is called the characteristics of logarithm of the number.

(ii) a decimal part which is always positive.

The integral part is called the **characteristic** and the decimal part is called the **mantissa**, of the logarithm of a number

(i) Characteristic of Logarithm of a Number > 1:

The above table shows that if number has one digit in the integral part, then the characteristic is zero; if its integral part has two digits, then the characteristics is one; with three digits in the integral part, the characteristic is two, and so on.

In other words, the characteristic of the logarithm of a number greater than 1 is always one less than the number of digits in the integral part of the number.

When a number 'b' is written in the scientific notation, i.e., in the form $b = a \times 10^n$ where $1 \leq a < 10$, the power of 10 i.e., n will give the characteristic of log b.

Examples:

Number	Scientific Notation	Characteristic of the Logarithm
1.02	1.02×10^0	0
99.6	9.96×10^1	1
102	1.02×10^2	2
1662.4	1.6624×10^3	3

(ii) Characteristic of Logarithm of a Number < 1:

The second part of the table indicates that, if a number has no zero immediately after the decimal point, the characteristic is -1; if has one zero immediately after the decimal point, the characteristic is -2; if it has two zeros immediately after the decimal point, the characteristics is -3; etc.

Example: Write the characteristic of the log of following numbers by expressing them in scientific notation and noting the power of 10.

Solution:

Number	Scientific Notation	Characteristic of the Logarithm
0.872	8.72×10^{-1}	-1
0.02	2.0×10^{-2}	-2
0.00345	3.45×10^{-3}	-3

When a number is less than 1, the characteristic of its logarithm is written by convention, as $\bar{3}$, $\bar{2}$ or $\bar{1}$ instead of -3, -2 or -1 respectively ($\bar{3}$ is read as bar 3) to avoid the mantissa becoming negative.

Remember: $\bar{2}.3748$ does not mean -2.3748. In $\bar{2}.3748$, 2 is negative but .3748 is positive; whereas in -2.3748 both 2 and .3748 are negative.

(c) Finding the mantissa of the Logarithm of a Number:

While the characteristic of the logarithm of a number is written merely by

inspection, the mantissa is found by making use of logarithmic tables. These tables have been constructed to obtain the logarithms up to 7 decimal places. For all practical purposes, a four-figure logarithmic table will provide sufficient accuracy.

A logarithmic table is divided into 3 parts.

(i) The first part of the table is the extreme left column headed by blank square. This column contains numbers from 10 to 99 corresponding to the first two digits of the number whose logarithm is required.

(ii) The second part of the table consist of 10 columns, headed by 0, 1, 2, ..., 9. These headings correspond to the third digit from the left of the number. The numbers under these columns record mantissa of the logarithms with decimal point omitted for simplicity.

(iii) The third part of the table further consists of a small columns know as mean differences columns headed by 1, 2, 3,...., 9. These headings correspond to the fourth digit the left of the number. The reading of these columns are added to the mantissa recorded in second part (ii) above.

When the four-figure log table is used to find the mantissa of the logarithm of a number, the decimal point is ignored and the number is rounded to four significant figures.

Using Tables to find log of a Number:

The method to find log of a number is explained in the following examples.

Example-1: Find the mantissa of the logarithm of 43.254.

Solution: Rounding off 43.254 we consider only the four significant digits 4325.

We first locate the row corresponding to 43 in the log tables and

i. Proceed horizontally till we reach the column corresponding to 2. The number at the intersection is 6355.
ii. Again proceeding horizontally till the mean difference column corresponding to 5 intersects this row, we get the number 6 at the intersection.
iii. Adding the two numbers 6355 and 5 we get .6360 as the mantissa of the logarithm of 43.25.

Example-2: Find the mantissa of the logarithm of 0.002347.

Solution: Here also, we consider only the four significant digits 2347.

We first locate the row corresponding to 23 in the logarithm tables and proceed as before.

Along the same row to its intersection with the column corresponding to 4 the resulting number is 36 92. The number at the intersection of this row and the mean difference column corresponding to 7 is 13. Hence the sum of 3692 and 13 gives the mantissa of the logarithm of 0.002347 as 0.3705.

Remember: The logarithms of numbers having the same sequence of significant digits have the same mantissa. e.g., the mantissa of numbers 0.002347 and 0.2347 is 0.3705

Finding the common logarithm of any given number:

(i) Round off the number to four significant digits.

(ii) Find the characteristic of the logarithm of the number by inspection.
(iii) Find the mantissa of the logarithm of the number from the log tables.
(iv) Combine the two.

Example-3: Find (i) log 278.23 (ii) log 0.07058

Solution:

(i) 278.23 can be round off as 278.2
The characteristics is 2 and the mantissa is .4443
So, $\log 278.23 = 2.4443.$

(ii) The characteristics of log 0.07058 is -2 which is written as $\bar{2}$ by convention. The mantissa is .8487.
So, $\log 0.0708 = \bar{2}.8487.$

The Concept of Antilogarithm and Use of Table:

The number whose logarithm is given is called antilogarithm.
i.e., if $\log_a y = x$, then y is the antilogarithm of x.

Finding the number whose Logarithm is known:

We ignore the characteristic and consider only the mantissa. In the antilogarithm page of the log table, we locate the row corresponding to the first two digits of the mantissa (taken together with the decimal point). Then we proceed along this row till it intersects the column corresponding to the third digit of the mantissa. The number at the intersection is added with the number at the intersection of this row and the mean difference column corresponding to the fourth digit of the mantissa.

Thus the significant figure of the required number is obtained. Now only the decimal point is to be fixed.

(i) If the characteristic of the given logarithm is positive, that number increased by 1 gives the number of figures to the left of the decimal point in the required number.
(ii) If the characteristic is negative, its numerical value decreased by 1 gives the number of zeros to the right of the decimal point in the required number.

Example: Find the numbers whose logarithms are

(i) 1.3247 (ii) $\bar{2}$.1324

Solution: (i) 1.3247

Reading along the row corresponding to .32, we get 2109 at the intersection of this row with the column corresponding to 4. The number at the intersection of this row and the mean difference column corresponding to 7 is 3. Adding 2109 and 3 we get 2112.

Since the characteristic is 1 it is increased by 1 (because there should be two digits in the integral part) and therefore the decimal point is fixed after two digits from left in 2112.

Hence antilog of 1.3247 is 21.12.

(ii) $\bar{2}$.1324

Proceeding as in (i) the significant figures corresponding to the mantissa are 1356.

Since the characteristics is $\bar{2}$, its numerical value 2 is decreased by 1. Hence there will be one zero after the decimal point.

Hence antilog of $\bar{2}.1324$ is 0.01356.

Solved Exercise 3.2

1. Find the common logarithms of the following numbers.

(i) 232.92

Solution: 232.92 can be round off as 232.9.
The characteristic is 2 and the mantissa is 0.3672
So, $\log(232.92) = 2.3672$

(ii) 29.326

Solution: 29.326 can be round off as 29.33.
The characteristic is 1 and the mantissa is 0.4673.
So, $\log(29.326) = 1.4673$

(iii) 0.00032

Solution: 0.00032 can be round off as 0.0003.
The characteristic is $\bar{4}$ and the mantissa is .5051.
So, $\log(0.00032) = \bar{4}.5051$.

(iv) 0.3206

Solution: 0.3206
The characteristic is $\bar{1}$ and the mantissa is 0.5059.
So, $\log(0.3206) = \bar{1}.5059$

2. If log 31.09 = 1.4926, find values of following:

(i) log 3.109

Solution: 3.109
The characteristic is 0 and the mantissa is 0.4926.
So, $\log(3.109) = 0.4926$

(ii) log 310.9

Solution: log 310.9
The characteristic is 2 and the mantissa is 0.4926.
So, $\log(310.9) = 2.4926$

(iii) log 0.003109

Solution: 0.003109 can be round off as 0.0031
The characteristic is $\bar{3}$ and the mantissa is 0.4926.
So, $\log(0.003109) = \bar{3}.4926$

(iv) log 0.3109

Solution: 0.3109
The characteristic is $\bar{1}$ and the mantissa is 0.4926.

So, $\log(0.3109) = \bar{1}.4926$

3. Find the numbers whose common logarithms are : (i) 3.5621 (ii) $\bar{1}.7427$

(i) Solution:

Let, logx be the number

So, $\log x = 3.5621$

Taking antilog on both sides

$x = \text{Antilog}(3.5621)$

Characteristic of $\log x = 3$

Mantissa of $\log x = 0.5621$

According to mantissa = 0.5621 (By using the table of antilogarithm)

$= 3648 + 1 = 3649$

$\Rightarrow$ $x = \text{Antilog}(3.5621)$

$x = 3649$

(ii) Solution:

Let x be the number

$\log x = \bar{1}.7427$

Taking antilog on both sides

$X = \text{antilog}(\bar{1}.7427)$

Characteristic of $\log x = 0.7457$

Mantissa of $\log x = 0.7427$

According to the mantissa = 0.7427 (By using the table of antilogarithm)

$= 5521 + 9 = 5530$

$\Rightarrow$ $x = \text{Antilog}(\bar{1}.7427)$

$X = 0.5530$

4. What replacement for the unknown in each of following will make the statement true?

(i) $\log_3 81 = L$

Solution: $\log_3 81 = L \Rightarrow 3^L = 81 \Rightarrow 3^L = 3^4$

Hence, $L = 4$

(ii) $\log_a 6 = 0.5$

Solution: $\log_a 6 = 0.5 \Rightarrow a^{0.5} = 6 \Rightarrow (a^{0.5})^2 = 6^2$

Hence, $a = 36$

(iii) $\log_5 n = 2$

Solution: $\log_5 n = 2 \Rightarrow 5^2 = n \Rightarrow 25 = n$

Hence, $n = 25$

(iv) $10^p = 40$

Solution: $10^p = 40 \Rightarrow \log_{10} 40 = p \Rightarrow p = 1.6021$

5. Evaluate

(i) $\log_2 \frac{1}{128}$

Solution: Let $\log_2 \frac{1}{128} = x \Rightarrow 2^x = \frac{1}{128} \Rightarrow 2^x = \frac{1}{2^7} \Rightarrow 2^x = 2^{-7}$

$\Rightarrow \quad x = -7$

(ii) log 512 to the base $2\sqrt{2}$.

Solution: Let $\log_{2\sqrt{2}} 512 = x$

Then its exponential form is $(2\sqrt{2})^x = 512$

i.e., $\left(2^1.2^{\frac{1}{2}}\right)^x = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

$$\left(2^{1+\frac{1}{2}}\right)^x = 2^9 \Rightarrow \left(2^{\frac{3}{2}}\right)^x = 2^9 \Rightarrow 2^{\frac{3x}{2}} = 2^9 \Rightarrow \frac{3x}{2} = 9$$

$$3x = 2 \times 9 \Rightarrow 3x = 18 \Rightarrow x = \frac{18}{3} \Rightarrow x = 6$$

6. Find the value of x from the following statements.

(i) $\log_2 x = 5$

Solution: $\log_2 x = 5 \Rightarrow 2^5 = x \Rightarrow x = 32$

(ii) $\log_{81} 9 = x$

Solution: $\log_{81} 9 = x \Rightarrow 81^x = 9 \Rightarrow (9^2)^x = 9 \Rightarrow 9^{2x} = 9^1$

$2x = 1 \Rightarrow x = \frac{1}{2}$

(iii) $\log_{64} 8 = \frac{x}{2}$

Solution: $\log_{64} 8 = \frac{x}{2} \Rightarrow 64^{\frac{x}{2}} = 8 \Rightarrow (8^2)^{\frac{x}{2}} = 8 \Rightarrow 8^x = 8^1$

$\Rightarrow x = 1$

(iv) $\log_x 64 = 2$

Solution: $\log_x 64 = 2 \Rightarrow x^2 = 64 \Rightarrow x^2 = 8^2 \Rightarrow x = 8$

(v) $\log_3 x = 4$

Solution: $\log_3 x = 4 \Rightarrow 3^4 = x \Rightarrow x = 81$

COMMON AND NATURAL LOGARITHM

We have introduced common logarithms having base 10. Common logarithms are also known as ***decadic logarithms*** named after its base 10. We usually take logx to mean $\log_{10}x$, and this type of logarithm is more convenient to use in numerical calculations. John Napier defined the logarithms tables to the base e. Napier's logarithms are also called ***Natural Logarithms***. He released the first ever log tables in 1614. $\log_e(x)$ is conventionally given the notation ln(x).

In many theoretical investigations in science and engineering, it is often convenient to have a base *e*, an irrational *n* umber, whose value is 2.7182818…..

Laws of Logarithm:

In this section we shall prove the laws of logarithm and then apply them to find products, quotients, powers and roots of numbers.

(i) $\log_a(mn) = \log_a m + \log_a n$ **(ii)** $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n\log_a m$ **(iv)** $\log_a n = \log_b n \times \log_a b$

Proofs of Laws of Logarithms:

(i) $\log_a(mn) = \log_a m + \log_a n$

Proof: Let $\log_a m = x$ and $\log_a n = y$

So in exponential form, we can write it as:

$a^x = m$ ____(i) and $a^y = n$ ____(ii)

Multiply eq. (i) and eq. (ii), we get

$a^x \times a^y = mn \Rightarrow a^{x+y} = mn \Rightarrow \log_a(mn) = x + y$

$\Rightarrow \log_a(mn) = \log_a m + \log_a n$

Hence Proved

(ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

Proof: Let $\log_a m = x$ and $\log_a n = y$

So in exponential form, we can write it as:

$a^x = m$ ____(i) and $a^y = n$ ____(ii)

Divide eq. (i) by eq. (ii), we get

$$\frac{a^x}{a^y} = \frac{m}{n} \Rightarrow a^{x-y} = \frac{m}{n} \Rightarrow \log_a\left(\frac{m}{n}\right) = x-y$$

$$\Rightarrow \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

(iii) $\log_a(m^n) = n\log_a m$

Proof: Let $\log_a m^n = x \Rightarrow a^x = m^n$

And $\log_a m = y \Rightarrow a^y = m$

Then $a^x = m^n = (a^y)^n = a^{yn}$

$\Rightarrow x = ny$

$\Rightarrow \log_a m^n = n\log_a m$

Hence Proved

(iv) Change of Base Formula

$\log_a n = \log_b n \times \log_a b$ OR $\log_a n = \frac{\log_b n}{\log_b a}$

Proof: Let $\log_b n = x \Rightarrow n = b^x$.

Taking "log" to the base 'a', we have

$\log_a n = \log_a b^x$

$= x\log_a b$

Thus $\log_a n = \log_b n \times \log_a b$ ________ (i)

Put $n = a$ in eq. (i), we get

$\log_a a = \log_b a \times \log_a b \Rightarrow 1 = \log_b a \times \log_a b$

$$\Rightarrow \log_a b = \frac{1}{\log_b a}$$

So, eq. (i) becomes

$$\log_a n = \log_b n \times \frac{1}{\log_b a}$$

$$\log_a n = \frac{\log_b n}{\log_b a}$$

Hence Proved

Example-1: Evaluate 291.3 × 42.36

Solution: Let $x = 291.3 \times 42.36$

Then $\log x = \log(291.3 \times 42.36)$

$= \log(9291.3) + \log(42.360)$

$= 2.4643 + 1.6269 = 4.0912$

$x = \text{Antilog}(4.0912) = 12340$

Example-2: Evaluate 0.2913 × 0.004236.

Solution: Let $y = 0.2913 \times 0.004236$

Then $\log y = \log(0.2913 \times 0.004236)$

$= \log(0.2913) + \log(0.0042360)$

$= \bar{1}.4643 + \bar{3}.6268$

$= \bar{3}.0912$ (**Note that:** $1 + \bar{1} + 3 = 1 - 1 - 3 = \bar{3}$)

$y = \text{Antilog}(\bar{3}.0911) = 0.001234$

Example-3: Evaluate $\frac{291.3}{42.36}$.

Solution: Let $x = \frac{291.3}{42.36}$

Then $\log x = \log \frac{291.3}{42.36}$

$\log x = \log(291.30) - \log(42.36) = 2.4643 - 1.6269 = 0.8374$

$x = \text{Antilog}(0.8374) = 6.877$

Example-4: Evaluate $\frac{0.002913}{0.04236}$

Solution: Let $y = \frac{0.002913}{0.04236}$

Then $\log y = \log(0.0029130 - \log(0.04236)$

$= \bar{3}.4643 - \bar{2}.6269 = \bar{3} + (0.4643 - 0.6269) - \bar{2}$

$= \bar{3} - 0.1626 - \bar{2}$

$= \bar{3} + (1 - 0.1626) - 1 - \bar{2}$ (adding and subtracting 1)

$= \bar{2}.8374$ $[\because 3 - 1 - 2 - = -3 - 1 - (-2) = -2 = 2]$

$y = \text{antilog}(\bar{2}.8374) = 0.06877$

Example-5: Evaluate $\sqrt[4]{(0.0163)^3}$

Solution: Let $y = \sqrt[4]{(0.0163)^3} = (0.0163)^{3/4}$

Then $\log y = \frac{3}{4}(\log 0.0163) = \frac{3}{4} \times \bar{2}.2122 = \frac{\bar{6}.6366}{4} = \frac{\bar{8} + 2.6366}{4}$

$= \bar{2} + 0.6592 = \bar{2}.6592$

$y = \text{Antilog}(\bar{2}.6592) = 0.04562$

Example-6: Calculate $\log_2 3 \times \log_3 8$

Solution: We know that

$$\log_a n = \frac{\log_b n}{\log_b a}$$

$$\because \quad \log_2 3 \times \log_3 8 = \frac{\log 3}{\log 2} \times \frac{\log 8}{\log 3} = \frac{\log 8}{\log 2} = \frac{\log 2^3}{3\log 2} = \frac{3\log 2}{\log 2} = 3$$

Solved Exercise 3.3

1. Write the following into sum or difference.

(i) $\log(A \times B)$

Solution: $\log(A \times B) = \log A + \log B$

(ii) $\log \frac{15.2}{30.5}$

Solution: $\log \frac{15.2}{30.5} = \log 15.2 - \log 30.5$

(iii) $\log \frac{21 \times 5}{8}$

Solution: $\log \frac{21 \times 5}{8} = \log 21 + \log 5 - \log 8$

(iv) $\log \sqrt[3]{\frac{7}{15}}$

Solution: $\log \sqrt[3]{\frac{7}{15}} = \log\left(\frac{7}{15}\right)^{\frac{1}{3}} = \frac{1}{3}\log\left(\frac{7}{15}\right) = \frac{1}{3}[\log 7 - \log 15]$

(v) $\log \frac{(22)^{\frac{1}{3}}}{5^3}$

Solution: $\log \frac{(22)^{1/3}}{5^3} = \log (22)^{\frac{1}{3}} - \log 5^3 = \frac{1}{3}\log 22 - 3\log 5.$

(vi) $\log \frac{25 \times 47}{29}$

Solution: $\log \frac{25 \times 47}{29} = \log 25 + \log 47 - \log 29$

2. Express $\log x - 2\log x + 3\log(x+1) - \log(x^2 - 1)$ as a single logarithm.

Solution: $\log x - 2\log x + 3\log(x+1) - \log(x^2 - 1)$.

$= \log x - \log x^2 + \log(x+1)^3 - \log(x^2 - 1)$.

$= \log x + \log(x+1)^3 - \log x^2 - \log(x^2 - 1)$

$$= \log \frac{x(x+1)^3}{x^2(x^2-1)} = \log \frac{x(x+1)^3}{x^2(x-1)(x+1)} = \log \frac{(x+1)^2}{x(x-1)}$$

3. Write the following in the form of a single logarithm.

(i) $\log 21 + \log 5$

Solution: $\log 21 + \log 5 = \log(21 \times 5)$

(ii) $\log 25 - 2\log 3$

Solution: $\log 25 - 2\log 3 = \log 25 - \log 3^2 = \log\frac{25}{3^2}$

(iii) $2\log x - 3\log y$

Solution: $2\log x - 3\log y = \log x^2 - \log y^3 = \log\frac{x^2}{y^3}$

(iv) $\log 5 + \log 6 - \log 2$

Solution: $\log 5 + \log 6 - \log 2 = \log\frac{5\times 6}{2}$

4. Calculate the following:

(i) $\log_3 2 \times \log_2 81$

Solution: $\log_3 2 \times \log_2 81 = \frac{\log 2}{\log 3}\times\frac{\log 81}{\log 2}$ $\quad \because \log_a n = \frac{\log n}{\log a}$

$$= \frac{\log 81}{\log 3} = \frac{\log 3^4}{\log 3} = \frac{4\log 3}{\log 3} = 4$$

(ii) $\log_5 3 \times \log_3 25$

Solution: $\log_5 3 \times \log_3 25 = \frac{\log 3}{\log 5}\times\frac{\log 25}{\log 3}$ $\quad \because \log_a n = \frac{\log n}{\log a}$

$$= \frac{\log 25}{\log 5} = \frac{\log 5^2}{\log 5} = \frac{2\log 5}{\log 5} = 2$$

5. If $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, then find the values of the following:

(i) $\log 32$

Solution: $\log 32 = \log 2^5 = 5\log 2 = 5(0.3010) = 1.5050$

(ii) $\log 24$

Solution: $\log 24 = \log(2\times 2\times 2\times 3) = \log(2^3\times 3) = \log 2^3 + \log 3 = 3\log 2 + \log 3$

$= 3(0.3010) + 0.4771 = 0.9030 + 0.4771 = 1.3801$

(iii) $\log\sqrt{3\frac{1}{3}}$

Solution: $\log\sqrt{3\frac{1}{3}} = \log\left(3\frac{1}{3}\right)^{\frac{1}{2}} = \frac{1}{2}\log\frac{10}{3} = \frac{1}{2}[\log 10 - \log 3] = \frac{1}{2}[\log(5\times 2) - \log 3]$

$$= \frac{1}{2}[\log 5 + \log 2 - \log 3] = \frac{1}{2}[0.6990 + 0.3010 - 0.4771]$$

$$= \frac{1}{2}[0.5229] = 0.2615$$

(iv) $\log\frac{8}{3}$

Solution: $\log\frac{8}{3} = \log\frac{2^3}{3} = \log 2^3 - \log 3 = 3\log 2 - \log 3 = 3(0.3010) - 0.4771$

$= 0.9030 - 0.4771 = 0.4259$

(v) log 30

Solution: $\log 30 = \log(2\times 3\times 5) = \log 2 + \log 3 + \log 5$

$= 0.3010 + 0.4771 + 0.6990$

$= 1.4771$

OOOOOOOOOOOOO

Application of Laws of Logarithm in Numerical Calculations:

So far we have applied laws of logarithm to simple type of products, quotients, powers or roots of numbers. We now extend their application to more difficult examples to verify their effectiveness in simplification.

Example-1: Show that: $7\log\frac{16}{15} + 5\log\frac{25}{24} + \log\frac{81}{80} = \log 2$

Solution: L.H.S. $= 7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$

$= 7[\log 16 - \log 15] + 5[\log 25 - \log 24] + 3[\log 81 - \log 80]$

$= 7[\log 2^4 - \log(3\times 5)] + 5[\log 5^2 - \log(2^3\times 3)] + 3[\log 3^4 - \log(2^4\times 5)]$

$= 7(4\log 2 - \log 3 - \log 5] + 5[2\log 5 - 3\log 2 - \log 3] + 3[4\log 3 - 4\log 2 - \log 5]$

$= (28 - 15 - 12)\log 2 + (-7 - 5 + 12)\log 3 + (-7 + 10 - 3)\log 5$

$= \log 2 + 0 + 0$

$= \log 2 =$ R.H.S.

Hence Proved

Example-2: Evaluate: $\sqrt[3]{\frac{0.07921\times(18.99)^2}{(5.79)^4\times 0.9474}}$

Solution: Let

$$y = \sqrt[3]{\frac{0.07921\times(18.99)^2}{(5.79)^4\times 0.9474}} = \left(\frac{0.07921\times(18.99)^2}{(5.79)^4\times 0.9474}\right)^{1/3}$$

$$\log y = \frac{1}{3}\log\left(\frac{0.07921\times(18.99)^2}{(5.79)^4\times 0.9474}\right)$$

$$= \frac{1}{3}[\log\{0.07921\times(18.99)^2\} - \log((5.79)^4\times 0.9474\}]$$

$$= \frac{1}{3}[\log(0.079210 + 2\log(18.99) - 4\log(5.79) - \log(0.9474)]$$

$$= \frac{1}{3}[\bar{2}.8988 + 2(1.2786) - 4(0.7627) - \bar{1}.9765]$$

$$= \frac{1}{3}[\bar{2}.8988 + 2.5572 - 3.0508 - \bar{1}.9765]$$

$$= \frac{1}{3}[1.4560 - 3.0273] = \frac{1}{3}(\bar{2}.4287)$$

$$= \frac{1}{3}(\bar{3} + 1.4287) = \bar{1} + 0.4762 = \bar{1}.4762$$

$y = \text{Antilog}(\bar{1}.47620 = 0.2993$

Example-3: Given $A = A_o e^{-kd}$. If $k = 2$, what should be the value of d to make $A = \frac{A_o}{2}$?

Solution: Given that $A = A_o e^{-kd}$ $\Rightarrow$ $\frac{A}{A_o} = e^{-kd}$

Substituting $k = 2$, and $A = \frac{A_o}{2}$, we get $\frac{1}{2} = e^{-2d}$

Taking common log on both sides, we get

$$\log_{10}1 - \log_{10}2 = -2d \log_{10}e$$

$$0 - 0.3010 = -2d\,(0.4343)$$

$$d = \frac{0.3010}{2\times 0.4343} = 0.3465$$

Solved Exercise 3.4

1. Use log tables to find the value of

(i) 0.8176×13.64

Solution: Let $y = 0.8176 \times 13.64$

$$\log y = \log(0.8176\times 13.64) = \log(0.8176) + \log(13.64)$$

$$= -1.9125 + 1.1348 = -0.0875 + 1.1348 = 1.0473$$

$$y = \text{Antilog}\,(1.0473) = 11.15$$

(ii) $(789.5)^{1/8}$

Solution: Let $y = (789.5)^{1/8}$

$$\log y = \log(789.5)^{\frac{1}{8}} = \frac{1}{8}\log(789.5) = \frac{1}{8}\times(2.8974) = 0.3622$$

$$y = \text{Anti}\log(0.3622) \quad y = 2.302$$

(iii) $\frac{0.678\times 9.01}{0.0234}$

Solution: Let $y = \frac{0.678\times 9.01}{0.0234}$

$$\log y = \log(0.678) + \log(9.01) - \log(0.0234)$$

$$= -1.8312 + 0.9542 - 2.3692 = (-1 + 0.8312) + 0.9542 - (-2 + 0.3692)$$

$$= -0.1688 + 0.9542 - (-1.6308) = -0.1688 + 0.9542 + 1.6308 = 2.4162$$

$$y = \text{Antilog}\,(2.4162) = 261$$

(iv) $\sqrt[5]{2.709} \times \sqrt[7]{1.239}$

Solution: Let $y = \sqrt[5]{2.709}\times\sqrt[7]{1.239}$

$$\log y = \log\left(\sqrt[5]{2.709} \times \sqrt[7]{1.239}\right) = \log(2.709)^{\frac{1}{5}} + \log(1.239)^{\frac{1}{7}}$$

$$= \frac{1}{5}\log(2.709) + \frac{1}{7}\log(1.239) = \frac{1}{5}(0.4328) + \frac{1}{7}(0.0931)$$

$$= 0.0866 + 0.0133 = 0.0999$$

$$y = \text{Anti}\log(0.0999) = 1.258$$

(v) $\dfrac{(1.23)(0.6975)}{(0.0075)(1278)}$

Solution: Let $y = \dfrac{(1.23)(0.6975)}{(0.0075)(1278)}$

$$\log y = \log \frac{(1.23)(0.6975)}{(0.0075)(1278)}$$
$$= \log(1.23) + \log(0.6975) - \log(0.0075) - \log(1278)$$
$$= 0.0899 + \bar{1}.8435 - \bar{3}.8751 - 3.1065$$
$$= 0.0899 + (-1 + .8435) - (-3 + 0.8751) - 3.1065$$
$$= 0.0899 - 0.1565 + 2.1249 - 3.1065$$
$$= -1.0482 = (2 - 1.0482) - 2 = \bar{2}.9518$$
$$y = \text{Anti}\log(\bar{2}.9518) = 0.08950$$

(vi) $\sqrt[3]{\dfrac{0.7214 \times 20.37}{60.8}}$

Solution: Let $y = \sqrt[3]{\dfrac{0.7214 \times 20.37}{60.8}}$

$$\log y = \log\left(\frac{0.7214 \times 20.37}{60.8}\right)^{\frac{1}{3}} = \frac{1}{3}\log\left(\frac{0.7214 \times 20.37}{60.8}\right)$$
$$= \frac{1}{3}\left[\log(0.7214) + \log(20.37) - \log(60.8)\right]$$
$$= \frac{1}{3}\left[\bar{1}.8582 + 1.3090 - 1.7839\right] = \frac{1}{3}\left[-1 + 0.8582 + 1.3090 - 1.7839\right]$$
$$= \frac{1}{3}\left[-0.1418 + 1.3090 - 1.7839\right] = -0.2056 = (1 - 0.2056) - 1$$
$$= \bar{1}.7944$$
$$y = \text{Antilog}(\bar{1}.7944) = 0.6229$$

(vii) $\dfrac{83 \times \sqrt[3]{92}}{127 \times \sqrt[5]{246}}$

Solution: Let $y = \dfrac{83 \times \sqrt[3]{92}}{127 \times \sqrt[5]{246}}$

$$\log y = \log \frac{83 \times \sqrt[3]{92}}{127 \times \sqrt[5]{246}}$$
$$\log y = \log(83) + \log(92)^{\frac{1}{3}} - \log(127) - \log(246)^{\frac{1}{5}}$$
$$\log y = \log(83) + \frac{1}{3}\log(92) - \log(127) - \frac{1}{5}\log(246)$$
$$\log y = 1.9191 + \frac{1}{3}(1.9638) - 2.1038 - \frac{1}{5}(2.3909)$$

$$\log y = 1.9191 + 0.6546 - 2.1038 - 0.4782$$
$$\log y = -0.0083$$
$$\log y = (1-0.0083) - 1$$
$$\log y = \bar{1}.9917$$
$$y = \text{Antilog}\,(\bar{1}.9917)$$
$$y = 0.9811$$

(viii) $\dfrac{(438)^3\sqrt{0.056}}{(388)^4}$

Solution: Let $y = \dfrac{(438)^3\sqrt{0.056}}{(388)^4}$

$$\log y = \log\frac{(438)^3(0.056)^{\frac{1}{2}}}{(388)^4}$$
$$\log y = \log(438)^3 + \frac{1}{2}\log(0.056) - \log(388)^4$$
$$\log y = 3\log(438) + \frac{1}{2}\log(0.056) - 4\log(388)$$
$$\log y = 3(2.6415) + \frac{1}{2}(\bar{2}.7482) - 4(2.5888)$$
$$\log y = 7.9245 + \frac{1}{2}(-2+0.7482) - 10.3552$$
$$\log y = 7.9245 + \frac{1}{2}(-1.2518) - 10.3552$$
$$\log y = 7.9245 - 0.6259 - 10.3552$$
$$\log y = -3.0566$$
$$\log y = (4-3.0566) - 4$$
$$\log y = \bar{4}.9434$$
$$y = \text{Antilog}\,(\bar{4}.9434) \qquad y = 0.0008778$$

2. A gas is expanding according to the law $pv^n = C$. Find C when $p=80$, $v=3.1$ and $n=\frac{5}{4}$.

Solution:
$$pv^n = C$$
$$\log(pv^n) = \log C$$
$$\log p + \log v^n = \log C$$
$$\log p + n\log v = \log C$$
$$\log(80) + \frac{5}{4}\log(3.1) = \log C$$
$$1.9030 + \frac{5}{4}\log(3.1) = \log C$$
$$1.9030 + \frac{5}{4}(0.4914) = \log C$$
$$1.9030 + 0.6143 = \log C$$

$$2.5173 = \log C$$

or
$$C = \text{Antilog}(2.5173)$$
$$C = 329.2$$

3. The formula $p = 90(5)^{-q/10}$ applies to the demand of a product, where q is the number of units and p is the price of one unit. How many units will be demanded if the price is Rs. 18.00?

Solution:
$$p = 90(5)^{-\frac{q}{10}}$$
$$\log p = \log\left(90(5)^{-\frac{q}{10}}\right)$$
$$\log p = \log 90 + \log(5)^{-\frac{q}{10}}$$
$$\log p = \log 90 + \left(-\frac{q}{10}\right)\log(5)$$
$$\log p = \log 90 - \frac{q}{10}\log 5$$
$$\log 18 = \log 90 - \frac{q}{10}\log 5$$
$$1.2553 = 1.9542 - \frac{q}{10}(0.6990)$$
$$1.2553 - 1.9542 = -q(0.6990)$$
$$-0.6989 = -\frac{q}{10}(0.6990)$$
$$q = \frac{0.06989}{0.6990}$$
$$q = 10\text{units}$$

4. If $A = \pi r^2$, find A, when $\pi = \frac{22}{7}$ and $r = 15$.

Solution:
$$A = \pi r^2$$
$$\log A = \log(\pi r^2)$$
$$\log A = \log\pi + \log r^2$$
$$\log A = \log\pi + 2\log r$$
$$\log A = \log\left(\frac{22}{7}\right) + 2\log(15)$$
$$\log A = 0.4973 + 2(1.1761)$$
$$\log A = 0.4973 + 2.3522$$
$$\log A = 2.8495$$
$$A = \text{Anti}\log(2.8495)$$
$$A = 707.1$$

5. If $V = \frac{1}{3}\pi r^2 h$, find V, when $\pi = \frac{22}{7}$, $r = 2.5$ and $h = 4.2$

Solution: $V = \frac{1}{3}\pi r^2 h$

$$\log V = \log\left(\frac{1}{3}\pi r^2 h\right)$$

$$\log V = \log\left(\frac{1}{3}\right) + \log\pi + \log r^2 + \log h$$

$$\log V = \log\left(\frac{1}{3}\right) + \log\pi + 2\log r + \log h$$

$$\log V = \log\left(\frac{1}{3}\right) + \log\left(\frac{22}{7}\right) + 2\log(2.5) + \log(4.2)$$

$$\log V = -0.4771 + 0.4973 + 2(0.3979) + 0.6232$$

$$\log V = -0.4771 + 0.4973 + 0.7958 + 0.6232$$

$$\log V = 1.4392$$

$$V = \text{Antilog}(1.4392)$$

$$V = 27.49$$

Solved Review Exercise 3

1. Multiple Choice Questions. Choose the correct answer.

(i) If $a^x = n$, then ________

(a) $a = \log_x n$ (b) $x = \log_n a$ (c) $x = \log_a n$ (d) $a = \log_n x$

(ii) The relation $y = \log_z x$ implies ________

(a) $x^y = z$ (b) $z^y = x$ (c) $x^z = y$ (d) $y^z = x$

(iii) The logarithm of unity to any base is ________

(a) 1 (b) 10 (c) e (d) 0

(iv) The logarithm of any number to itself as base is ________

(a) 1 (b) 0 (c) -1 (d) 10

(v) $\log e$ = ________, where $e \approx 2.718$

(a) 0 (b) 0.4343 (c) ∞ (d) 1

(vi) The value of $\log\left(\frac{p}{q}\right)$ is ________

(a) $\log p - \log q$ (b) $\frac{\log p}{\log q}$ (c) $\log p + \log q$ (d) $\log q - \log p$

(vii) $\log p - \log q$ is same as ________

(a) $\log\left(\frac{q}{p}\right)$ (b) $\log(p-q)$ (c) $\frac{\log p}{\log q}$ (d) $\log\left(\frac{p}{q}\right)$

(viii) $\log(m^n)$ can be written as ________

(a) $(\log m)^n$ (b) $m \log n$ (c) $n \log m$ (d) $\log(mn)$

(ix) $\log_b a \times \log_c b$ can be written as ________

(a) $\log_a c$ (b) $\log_c a$ (c) $\log_a b$ (d) $\log_b c$

(x) $\log_y x$ will be equal to ________

(a) $\frac{\log_z x}{\log_y z}$ (b) $\frac{\log_x z}{\log_y z}$ (c) $\frac{\log_z x}{\log_z y}$ (d) $\frac{\log_z y}{\log_z x}$

Answers: (i) c (ii) b (iii) d (iv) a (v) b
(vi) a (vii) d (viii) c (ix) b (x) c

2. Complete the following.

(i) For common logarithms, the base is ....

(ii) The integral part of the common logarithm of a number is called the ....

(iii) The decimal part of the common logarithm of a number is called the ....

(iv) If $x = \log y$, then y is called the ... . of x.

(v) If the characteristic of the logarithm of a number is $\bar{2}$, that number will have .... zero(s) immediately after the decimal point.

(vi) If the characteristic of the logarithm of a number is 1, that number will have .... digits in its integral part.

Answers: (i) 10 (ii) characteristic (iii) mantissa
(iv) anti-logarithm (v) one (vi) 2

3. Find the value of x in the following.

(i) $\log_3 x = 5$

Solution: $\log_3 x = 5 \Rightarrow x = 3^5 \Rightarrow x = 243$

(ii) $\log_4 256 = x$

Solution: $\log_4 256 = x \Rightarrow 4^x = 256 \Rightarrow 4^x = 4^4 \Rightarrow x = 4$

(iii) $\log_{625} 5 = \frac{1}{4}x$

Solution: $\log_{625} 5 = \frac{1}{4}x \Rightarrow 625^{\frac{1}{4}x} = 5 \Rightarrow (5^4)^{\frac{1}{4}x} = 5 \Rightarrow 5^x = 5^1$
$\Rightarrow x = 1$

(iv) $\log_{64} x = -\frac{2}{3}$

Solution: $\log_{64} x = -\frac{2}{3} \Rightarrow x = 64^{-\frac{2}{3}} \Rightarrow x = 4^{3\left(-\frac{2}{3}\right)} \Rightarrow x = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

4. Find the value of x in the following.

(i) $\log x = 2.4543$

Solution: $\log x = 2.4543$
$x = \text{Anti}\log(2.4543) = 284.6$

(ii) $\log x = 0.1821$

Solution:
$\log x = 0.1821$
$x = \text{Anti}\log(0.1821) = 1.521$

(iii) $\log x = 0.0044$

Solution:
$\log x = 0.004$
$x = \text{Anti}\log(0.004) = 1.010$

(iv) $\log x = \bar{1}.6238$

Solution: $\log x = \bar{1}.6238$

$$x = \text{Anti}\log(\bar{1}.6238) = 0.04205$$

5. If log 2 = 0.3010, log 3 = 0.4771 and log 5 = 0.6990, then find the values of the following.

(i) log 45

Solution:

$$\begin{aligned}\log 45 &= \log(3\times3\times5)\\ &= \log3+\log3+\log5\\ &= 0.4771+0.4771+0.6990\\ &= 1.6532\end{aligned}$$

(ii) $\log\frac{16}{15}$

Solution:

$$\begin{aligned}\log\frac{16}{15} &= \log16-\log15\\ &= \log2^4-\log(3\times5)\\ &= 4\log2-[\log3+\log5]\\ &= 4\log2-\log3-\log5\\ &= 4(0.3010)-0.4771-0.6990\\ &= 1.2040-0.4771-0.6990\\ &= 0.0279\end{aligned}$$

(iii) log 0.048

Solution:

$$\begin{aligned}\log 0.048 &= \log\frac{48}{1000}\\ &= \log48-\log1000\\ &= \log(2\times2\times2\times2\times3)-\log(2\times2\times2\times5\times5\times5)\\ &= \log(2^4\times3)-\log(2^3\times5^3)\\ &= [\log2^4+\log3]-[\log2^3+\log5^3]\\ &= 4\log2+\log3-3\log2-3\log5\\ &= 4(0.3010)+0.4771-3(0.3010)-3(0.6990)\\ &= 1.204+0.4771-0.903-2.097\\ &= -1.3189\\ &= (2-1.3189)-2\\ &= \bar{2}.6811\end{aligned}$$

6. Simplify the following.

(i) $\sqrt[3]{25.47}$

Solution: Let $y = \sqrt[3]{25.47}$

$$\log y = \log(25.47)^{\frac{1}{3}}$$

$$\log y = \frac{1}{3}\log(25.47)$$

$$\log y = \frac{1}{3}(1.4060)$$

$$\log y = 0.4687$$

$$y = \text{Anti}\log(0.4687)$$

$$y = 2.942$$

(ii) $\sqrt[5]{342.2}$

Solution: $y = \sqrt[5]{342.2}$

$$\log y = \log(342.2)^{\frac{1}{5}}$$

$$\log y = \frac{1}{5}\log(342.2)$$

$$\log y = \frac{1}{5}(2.5343)$$

$$\log y = 0.5069$$

$$y = \text{Anti}\log(0.5069) \quad y = 3.213$$

(iii) $\dfrac{(8.97)^3 \times (3.95)^2}{\sqrt[3]{15.37}}$

Solution: Let $y = \dfrac{(8.97)^3 \times (3.95)^2}{\sqrt[3]{15.37}}$

$$\log y = \log(8.97)^3 + \log(3.95)^2 - \log(15.37)^{\frac{1}{3}}$$

$$\log y = 3\log(8.97) + 2\log(3.95) - \frac{1}{3}\log(15.37)$$

$$\log y = 2.8584 + 1.1932 - 0.3956$$

$$\log y = 3.6560$$

$$y = \text{Anti}\log(3.6560)$$

$$y = 4528.98 \Rightarrow 4529$$

SUMMARY

- If $a^x = y$, then x is called the **logarithm** of y to the base a and is written as $x = \log_a y$, where $a > 0$, $a \neq 1$ and $y > 0$.
- If $x = \log_a y$, then $a^x = y$.
- If the base of the logarithm is taken as 10, it is known as common logarithm and if

the base is taken as e ($\simeq$2.718) then it is known as natural or Naperian logarithm.

- The integral part of the common logarithm of a number is called the characteristic and the decimal part the mantissa.
- (i) For a number greater than 1, the characteristic of its logarithm is equal to the number of digits in the integral part of the number minus one.
  (ii) For a number less than 1, the characteristic is always negative and is equal to the number of zeros immediately after the decimal point of the number plus one.
- When a number is less than 1, the characteristic is always written as $\bar{3}$, $\bar{2}$, $\bar{1}$ (instead of $-3, -2, -1$) to avoid the mantissa becoming negative.
- The logarithms of numbers having the same sequence of significant digits have the same mantissa.
- The number corresponding to a given logarithm is known as antilogarithm.
- $\log_e 10 = 2.3026$ and $\log_{10} e = 0.4343$.
- Laws of logarithms:
  (i) $\log_a(mn) = \log_a m + \log_a n$
  (ii) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$
  (iii) $\log_a (m^n) = n \log_a m$
  (iv) $\log_a n = \log_b n \times \log_a b$

UNIT 4

ALGEBRAIC EXPRESSIONS AND ALGEBRAIC FORMULAS

Unit Outlines			
4.1	*Algebraic Expressions*	4.2	*Algebraic Formulae*
4.3	*Surds and their Application*	4.4	*Rationalization*

STUDENTS LEARNING OUTCOMES

After studying this unit the students will be able to:

- know that a rational expression behaves like a rational number.
- define a rational expression as the quotient $\frac{p(x)}{q(x)}$ of two polynomials *p(x)* and *q(x)* where *q(x) is* not the zero polynomial.
- examine whether a given algebraic expression is a
  - ★ polynomial or not
  - ★ rational expression or not
- define $\frac{p(x)}{q(x)}$ as a rational expression in its lowest terms if p*(x)* and *q(x)* are polynomials with integral coefficients and having no common factor.
- examine whether a given rational algebraic expression is in lowest form or not.
- reduce a given rational expression to its lowest terms.
- find the sum, difference and product of rational expressions.
- divide a rational expression with another and express the result in its lowest terms.
- find value of algebraic expression at some particular real number.
- know the formulas:

  $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

  $(a + b)^2 - (a - b)^2 = 4ab$
- find the value of $a^2 + b^2$ and of ab when the values of $a + b$ and $a - b$ are known.
- know the formula:

  $(a + b + c)^2 = a^2+b^2+c^2+2ab+ 2bc + 2ca.$
- find the value of $a^2+b^2+c^2$ when the values of $a+b+c$ and $ab+bc+ca$ are given.
  - ★ find the value of $a+b+c$ when the values of $a^2 + b^2 + c^2$ and $ab +bc+ ca$ are given.
  - ★ find the value of $ab+bc+ca$ when the values of $a^2+b^2+c^2$ and $a+b+c$ are given.

- Know the formulas
  $(a+b)^3 = a^3 + 3ab(a+b) + b^3$
  $(a-b)^3 = a^3 - 3ab(a-b) - b^3$
  - ★ Find the value of $a^3 \pm b^3$ when the values of $a \pm b$ and ab are given.
  - ★ Find the value of $x^3 \pm \frac{1}{x^3}$ when the value of $x \pm \frac{1}{x}$ is given.
- Know the formula
  $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.
  - ★ Find the product of $x+\frac{1}{x}$ and $x^2 + \frac{1}{x^2} - 1$.
  - ★ Find the product of $x - \frac{1}{x}$ and $x^2 + \frac{1}{x^2} + 1$.
  - ★ Find the continued product of
    $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.
- Recognize the surds and their application.
- Explain the surds of second order. Use basic operations on surds of second order to rationalize the denominators and evaluate it.
- Explain rationalization (with precise meaning) of real numbers of the types $\frac{1}{a+b\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations where x and y are natural numbers and a and b integers.

Algebraic Expressions:

Algebra is a generalization of arithmetic. Recall that when operations of addition and subtraction are applied to algebraic terms we obtain an algebraic expression. For instance, $5x^2 - 3x + \frac{2}{\sqrt{x}}$ and $3xy + \frac{3}{x}(x \neq 0)$ are algebraic expressions.

Polynomials:

A polynomial in the variable x is an algebraic expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0, \qquad a_n \neq 0, \quad \ldots\ldots(i)$$

Where n, the highest power of x, is a non-negative integer called the ***degree of the polynomial*** and each coefficient a_n is a real number. The coefficient a_n of the highest power of x is called the ***leading coefficient of the polynomial.***

From the study of similar properties of integers and polynomials w.r.t. addition and multiplication we may say that polynomials behave like integers.

Expressions Behave like Rational Number:

Let a and b be two integers, then $\frac{a}{b}$ is not necessarily an integer. Therefore, number system is extended and, is defined as a rational number where $a, b \in Z$ and $b \neq 0$.

Similarly, if $p(x)$ and $q(x)$ are two polynomials, then $\frac{p(x)}{q(x)}$ *is* not necessarily a

polynomial, where $q(x) \neq 0$. Therefore, similar to the idea of rational numbers, concept of rational expressions is developed.

Rational Expression:

The quotient $\frac{p(x)}{q(x)}$ *of* two polynomials $p(x)$ and $q(x)$, where $q(x)$ is a non-zero polynomial is called a ***rational expression. For example,*** $\frac{2x+1}{3x+8}, 3x+8 \neq 0$ is a rational expression.

In the rational expression $\frac{p(x)}{q(x)}$, $p(x)$ is called the ***numerator*** and $q(x)$ is known as the ***denominator*** of the rational expression $\frac{p(x)}{q(x)}$. The rational expression $\frac{p(x)}{q(x)}$ need not be a polynomial.

Remember: Every polynomial $p(x)$ can be regarded as a rational expression, since we can write $p(x)$ as $\frac{p(x)}{1}$. Thus, every polynomial is a rational expression, but every rational expression need not be a polynomial.

Properties of Rational Expressions:

The method for operations with rational expressions is similar to operations with rational numbers.

Let $p(x)$, $q(x)$, $r(x)$, $s(x)$ be any polynomials such that all values of the variable that make a rational expression undefined are excluded from the domain.

Then following properties of rational expressions hold under the supposition that they all are defined (i.e., denominator (s) $\neq 0$)

(i) $\frac{p(x)}{q(x)} = \frac{r(x)}{s(x)}$ if and only if $p(x)\,s(x) = q(x)\,r(x)$. (Equality)

(ii) $\frac{p(x)k}{q(x)k} = \frac{p(x)}{q(x)}$ (Cancellation)

(iii) $\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x)+q(x)r(x)}{q(x)s(x)}$ (Addition)

(iv) $\frac{p(x)}{q(x)} - \frac{r(x)}{s(x)} = \frac{p(x)s(x)-q(x)r(x)}{q(x)s(x)}$ (Subtraction)

(v) $\frac{p(x)}{q(x)} \cdot \frac{r(x)}{s(x)} = \frac{p(x)r(x)}{q(x)s(x)}$ (Multiplication)

(vi) $\frac{p(x)}{q(x)} \div \frac{r(x)}{s(x)} = \frac{p(x)s(x)}{q(x)r(x)}$ (Division)

(vii) Additive inverse of $\frac{p(x)}{q(x)}$ is "$-\frac{p(x)}{q(x)}$".

(viii) Multiplicative inverse or reciprocal of $\frac{p(x)}{q(x)}$ is $\frac{q(x)}{p(x)}$, $p(x)\neq 0$, $q(x)\neq 0$.

Rational Expression in its Lowest form:

The rational expression $\frac{p(x)}{q(x)}$ is said to be in its lowest forms if *p(x)* and *q(x)* are polynomials with integral coefficients and have no common factor.

For example, $\frac{x+1}{x^2+1}$ is in its lowest form.

To examine whether a rational expression is in lowest forms or not:

To examine the rational expression $\frac{p(x)}{q(x)}$, find H.C.F. of *p(x)* and *q(x)* if H.C.F is 1 then the rational expression is in lowest form. **For example,** $\frac{x-1}{x^2+1}$ is in its lowest form as H.C.F *of* $x-1$ *and* x^2+1 *is* 1.

Working Rule to reduce a rational expression to its lowest forms:

Let the given rational expression be $\frac{p(x)}{q(x)}$.

Step I: Factorize each of the two polynomials *p(x)* and *q(x)*.
Step II: Find HCF of *p(x)* and *q(x)*.
Step III: Divide the numerator *p(x)* and the denominator *q(x)* by the H.C.F *of p(x)*and *q(x)*. The rational expression so obtained, is in its lowest forms.

In other, words an algebraic fraction can be reduced to its lowest forms by first factorizing both the polynomials in the numerator and the denominator and then canceling the common factors between them.

Example: Reduce the following algebraic fractions to their lowest forms.

(i) $\frac{lx+mx-ly-my}{3x^2-3y^2}$ **(ii)** $\frac{3x^2+18x+27}{5x^2-45}$

Solution:

(i) $\frac{lx+mx-ly-my}{3x^2-3y^2}=\frac{x(l+m)-y(l+m)}{3(x^2-y^2)}=\frac{(l+m)(x-y)}{3(x+y)(x-y)}=\frac{l+m}{3(x+y)}$

(ii) $\frac{3x^2+18x+27}{5x^2-45}=\frac{3(x^2+6x+9)}{5(x^2-9)}=\frac{3(x+3)(x+3)}{5(x+3)(x-3)}=\frac{3(x+3)}{5(x-3)}$

Sum, Difference and Product of Rational Expressions:

For finding sum and difference of algebraic expressions containing rational

expressions, we take the L.C.M. of the denominators and simplify as explained in the following examples.

Example-1: Simplify (i) $\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{x^2-y^2}$ **(ii)** $\frac{2x^2}{x^4-16}-\frac{x}{x^2-4}+\frac{1}{x+2}$

Solution:

(i) $$\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{x^2-y^2}=\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{(x+y)(x-y)}$$

$$=\frac{x+y-(x-y)+2x}{(x+y)(x-y)}=\frac{x+y-x+y+2x}{(x+y)(x-y)}$$

$$=\frac{2x+2y}{(x+y)(x-y)}=\frac{2(x+y)}{(x+y)(x-y)}=\frac{2}{x-y}$$

(ii) $$\frac{2x^2}{x^4-16}-\frac{x}{x^2-4}+\frac{1}{x+2}=\frac{2x^2}{(x^2+4)(x^2-4)}-\frac{x}{x^2-4}+\frac{1}{x+2}$$

$$=\frac{2x^2}{(x^2+4)(x+2)(x-2)}-\frac{x}{(x+2)(x-2)}+\frac{1}{x+2}$$

$$=\frac{2x^2-x(x^2+4)+(x^2+4)(x-2)}{(x^2+4)(x+2)(x-2)}$$

$$=\frac{2x^2-x^3-4x+x^3+4x-2x^2-8}{(x^2+4)(x+2)(x-2)}$$

$$=\frac{-8}{(x^2+4)(x+2)(x-2)}$$

$$=\frac{-8}{(x^2+4)(x^2-4)}=\frac{-8}{x^4-16}$$

Example-2: Find the product $\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}$ **(in simplified form)**

Solution: $$\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}=\frac{(x+2)\left[(2x)^2-(3y)^2\right]}{(2x-3y)(x+2)y}$$

$$=\frac{(x+2)(2x+3y)(2x-3y)}{y(x+2)(2x-3y)}=\frac{2x+3y}{y}$$

Dividing a Rational Expression with another Rational Expression:

In order to divide one rational expression with another, we first invert for changing

division to multiplication and simplify the resulting product to the lowest terms.

Example: Simplify $\dfrac{7xy}{x^2-4x+4} \div \dfrac{14y}{x^2-4}$

Solution: $\dfrac{7xy}{x^2-4x+4} \div \dfrac{14y}{x^2-4} = \dfrac{7xy}{x^2-4x+4} \cdot \dfrac{x^2-4}{14y}$

$$= \frac{7xy}{(x-2)(x-2)} \cdot \frac{(x+2)(x-2)}{14y} = \frac{x(x+2)}{2(x-2)}$$

Evaluation of Algebraic Expression at some particular Real Number:

Definition: If specific numbers are substituted for the variables in an algebraic expression, the resulting number is called the ***value of the expression.***

Example: Evaluate $\dfrac{3x^2\sqrt{y}+6}{5(x+y)}$ **if x = -4 and y=9.**

Solution: $\dfrac{3x^2\sqrt{y}+6}{5(x+y)} = \dfrac{3(-4)^2\sqrt{9}+6}{5(-4+9)} = \dfrac{3(16)(3)+6}{5(5)} = \dfrac{150}{25} = 6$

Solved Exercise 4.1

1. Identify whether the following algebraic expressions are polynomials (Yes or No).

(i) $3x^2+\dfrac{1}{x}-5$ (ii) $3x^3-4x^2-x\sqrt{x}+3$ (iii) $x^2-3x+\sqrt{2}$ (iv) $\dfrac{3x}{2x-1}+8$

Ans: (i) No (ii) No (iii) Yes (iv) No

2. State whether each of the following expression is a rational expression or not.

(i) $\dfrac{3\sqrt{x}}{3\sqrt{x}+5}$ (ii) $\dfrac{x^3-2x^2+\sqrt{3}}{2+3x-x^2}$ (iii) $\dfrac{x^2+6x+9}{x^2-9}$ (iv) $\dfrac{2\sqrt{x}+3}{2\sqrt{x}-3}$

Ans: (i) No (ii) Yes (iii) Yes (iv) No

3. Reduce the following rational expressions to the lowest form.

(i) $\dfrac{120x^2y^3z^5}{30x^3yz^2}$

Solution: $\dfrac{120x^2y^3z^5}{30x^3yz^2} = 4x^2y^3z^5 \times x^{-3}y^{-1}z^{-2} = 4x^{2-3}y^{3-1}z^{5-2} = 4x^{-1}y^2z^3 = \dfrac{4y^2z^3}{x}$

(ii) $\dfrac{8a(x+1)}{2(x^2-1)}$

Solution: $\dfrac{8a(x+1)}{2(x^2-1)} = \dfrac{4a(x+1)}{(x+1)(x-1)} = \dfrac{4a}{x-1}$

(iii) $\dfrac{(x+y)^2-4xy}{(x-y)^2}$

Solution: $\dfrac{(x+y)^2-4xy}{(x-y)^2}=\dfrac{x^2+y^2+2xy-4xy}{x^2+y^2-2xy}=\dfrac{x^2+y^2-2xy}{x^2+y^2-2xy}=1$

(iv) $\dfrac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$

Solution: $\dfrac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}=\dfrac{(x^3-y^3)(x^2-2xy+y^2)}{(x^3-y^3)}$

$=x^2+y^2-2xy=(x-y)^2$

(v) $\dfrac{(x+2)(x^2-1)}{(x+1)(x^2-4)}$

Solution: $\dfrac{(x+2)(x^2-1)}{(x+1)(x^2-4)}=\dfrac{(x+2)(x+1)(x-1)}{(x+1)(x+2)(x-2)}=\dfrac{(x-1)}{(x-2)}$

(vi) $\dfrac{x^2-4x+4}{2x^2-8}$

Solution: $\dfrac{x^2-4x+4}{2x^2-8}=\dfrac{(x-2)^2}{2(x^2-4)}=\dfrac{(x-2)(x-2)}{2(x+2)(x-2)}=\dfrac{x-2}{2(x+2)}$

(vii) $\dfrac{64x^5-64x}{(8x^2+8)(2x+2)}$

Solution: $\dfrac{64x^5-64x}{(8x^2+8)(2x+2)}=\dfrac{64x(x^4-1)}{8(x^2+1)2(x+1)}=\dfrac{64x(x^2+1)(x^2-1)}{16(x^2+1)(x+1)}$

$=\dfrac{4x(x+1)(x-1)}{(x+1)}=4x(x-1)$

(viii) $\dfrac{9x^2-(x^2-4)^2}{4+3x-x^2}$

Solution: $\dfrac{9x^2-(x^2-4)^2}{4+3x-x^2}=\dfrac{(3x)^2-(x^2-4)^2}{4+3x-x^2}=\dfrac{(3x-x^2+4)(3x+x^2-4)}{(3x-x^2+4)}$

$=(x^2+3x-4)$

4. (a) Evaluate (a) $\dfrac{x^3y-2z}{xz}$ ***for*** **(i)** $x=3, y=-1, z=-2$ **(ii)** $x=-1, y=-9, z=4$

Solution (i): $\dfrac{x^3y-2z}{xz}=\dfrac{(3)^3(-1)-2(-2)}{(3)(-2)}=\dfrac{(27)(-1)+4}{-6}=\dfrac{-23}{-6}=\dfrac{23}{6}=3\dfrac{5}{6}$

Solution (ii): $\frac{x^3y-2z}{xz}=\frac{(-1)^3(-9)-2(4)}{(-1)(4)}=\frac{9-8}{-4}=-\frac{1}{4}$

(b) $\frac{x^2y^3-5z^4}{xyz}$ **for x = 4, y = -2, z = -1.**

Solution: $\frac{x^2y^3-5z^4}{xyz}=\frac{(4)^2(-2)^3-5(-1)^4}{(4)(-2)(-1)}=\frac{16(-8)-5(1)}{8}$

$=\frac{-128-5}{8}=-\frac{133}{8}=-16\frac{5}{8}$

5. Perform the indicated operation and simplify.

(i) $\frac{15}{2x-3y}-\frac{4}{3y-2x}$

Solution: $\frac{15}{2x-3y}-\frac{4}{3y-2x}=\frac{15}{2x-3y}-\frac{4}{-(2x-3y)}=\frac{15}{2x-3y}+\frac{4}{2x-3y}$

$=\frac{15+4}{2x-3y}=\frac{19}{2x-3y}$

(ii) $\frac{1+2x}{1-2x}-\frac{1-2x}{1+2x}$

Solution: $\frac{1+2x}{1-2x}-\frac{1-2x}{1+2x}=\frac{(1+2x)^2-(1-2x)^2}{(1-2x)(1+2x)}=\frac{1+4x^2+4x-(1+4x^2-4x)}{(1)^2-(2x)^2}$

$=\frac{1+4x^2+4x-1-4x^2+4x}{1-4x^2}=\frac{8x}{1-4x^2}$

(iii) $\frac{x^2-25}{x^2-36}-\frac{x+5}{x+6}$

Solution: $\frac{x^2-25}{x^2-36}-\frac{x+5}{x+6}=\frac{(x^2-25)-(x+5)(x-6)}{x^2-36}$

$=\frac{(x+5)(x-5)-(x+5)(x-6)}{x^2-36}=\frac{(x+5)(x-5-x+6)}{x^2-36}$

$=\frac{(x+5)(1)}{x^2-36}=\frac{x+5}{x^2-36}$

(iv) $\frac{x}{x-y}-\frac{y}{x+y}-\frac{2xy}{x^2-y^2}$

Solution: $\frac{x}{x-y}-\frac{y}{x+y}-\frac{2xy}{x^2-y^2}$

$=\frac{x^2+xy-xy+y^2-2xy}{x^2-y^2}=\frac{x^2+y^2-2xy}{(x+y)(x-y)}=\frac{(x-y)^2}{(x+y)(x-y)}$

$$=\frac{(x-y)(x-y)}{(x+y)(x-y)}=\frac{x-y}{x+y}$$

(v) $\frac{x-2}{x^2+6x+9}-\frac{x+2}{2x^2-18}$

Solution: $\frac{x-2}{x^2+6x+9}-\frac{x+2}{2x^2-18}=\frac{x-2}{(x)^2+2(x)(3)+(3)^2}-\frac{x+2}{2(x+3)(x-3)}$

$$=\frac{x-2}{(x+3)^2}-\frac{x+2}{2(x+3)(x-3)}=\frac{2(x-3)(x-2)-(x+2)(x+3)}{2(x-3)(x+3)^2}$$

$$=\frac{2(x^2-2x-3x+6)-(x^2+3x+2x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2(x^2-5x+6)-(x^2+5x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2x^2-10x+12-x^2-5x-6}{2(x-3)(x+3)^2}=\frac{x^2-15x+6}{2(x-3)(x+3)^2}$$

(vi) $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$

Solution: $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$

$$=\frac{(x+1)(x^2+1)-(x-1)(x^2+1)-2(x^2-1)-4}{(x^4-1)}$$

$$=\frac{(x^3+x+x^2+1)-(x^3+x-x^2-1)-(2x^2-2)-4}{(x^4-1)}$$

$$=\frac{x^3+x^2+x+1-x^3+x^2-x+1-2x^2+2-4}{x^4-1}=\frac{0}{x^4-1}=0$$

6. Perform the indicated operation and simplify.

(i) $(x^2-49).\frac{5x+2}{x+7}$

Solution: $(x^2-49).\frac{5x+2}{x+7}=(x+7)(x-7).\frac{5x+2}{(x+7)}=(x-7)(5x+2)$

(ii) $\frac{4x-12}{x^2-9}+\frac{18-2x^2}{x^2+6x+9}$

Solution: $\frac{4x-12}{x^2-9}\div\frac{18-2x^2}{x^2+6x+9}$

$$=\frac{4(x-3)}{(x+3)(x-3)}\div\frac{2(9-x^2)}{(x)^2+2(x)(3)+(3)^2}$$

$$=\frac{4(x-3)}{(x+3)(x-3)}\times\frac{(x+3)^2}{2\left[(3)^2-(x)^2\right]}$$

$$=\frac{4(x-3)}{(x+3)(x-3)}\times\frac{(x+3)(x+3)}{2(3+x)(3-x)}=\frac{2}{(3-x)}$$

(iii) $\frac{x^6-y^6}{x^2-y^2}\div\left(x^4+x^2y^2+y^4\right)$

Solution: $\frac{x^6-y^6}{x^2-y^2}\div\left(x^4+x^2y^2+y^4\right)=\frac{(x^2)^3-(y^2)^3}{(x^2-y^2)}\times\frac{1}{(x^4+x^2y^2+y^4)}$

$$=\frac{(x^2-y^2)(x^4+x^2y^2+y^4)}{(x^2-y^2)}\times\frac{1}{(x^4+x^2y^2+y^4)}=1$$

(iv) $\frac{x^2-1}{x^2+2x+1}\cdot\frac{x+5}{1-x}$

Solution: $\frac{x^2-1}{x^2+2x+1}\cdot\frac{x+5}{1-x}=\frac{(x+1)(x-1)}{(x+1)^2}\cdot\frac{x+5}{-(x-1)}$

$$=\frac{(x+1)}{(x+1)(x+1)}\cdot-(x+5)=\frac{-(x+5)}{x+1}$$

(v) $\frac{x^2+xy}{y(x+y)}\cdot\frac{x^2+xy}{y(x+y)}\div\frac{x^2-x}{xy-2y}$

Solution: $\frac{x^2+xy}{y(x+y)}\cdot\frac{x^2+xy}{y(x+y)}\div\frac{x^2-x}{xy-2y}=\frac{x^2+xy}{y(x+y)}\cdot\frac{x^2+xy}{y(x+y)}\div\frac{x^2-x}{xy-2y}$

$$=\frac{x(x+y)}{y(x+y)}\cdot\frac{x(x+y)}{y(x+y)}\times\frac{y(x-2)}{x(x-1)}=\frac{x(x-2)}{y(x-1)}$$

ALGEBRAIC FORMULAE

Using the formulas:

(i) $(a+b)^2+(a-b)^2=2(a^2+b^2)$ *and* $(a+b)^2-(a-b)^2=4ab$

The process of finding the values of a^2+b^2 and ab is explained in the following examples.

Example: If $a+b=7$ and $a-b=3$, then find the value of (a) a^2+b^2 (b) ab

Solution: Given $a+b=7$ and $a-b=3$.

(a) To find the value of (a^2+b^2), we use the formula

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$

Substituting the values $a + b = 7$ and $a - b = 3$, we get

$$(7)^2 + (3)^2 = 2(a^2 + b^2)$$
$$49+9 = 2(a^2+b^2)$$
$$58 = 2(a^2 + b^2)$$
$$29 = a^2 + b^2$$

Or $\quad a^2 + b^2 = 29$

(b) To find the value of *ab*, we make use of the formula

$$(a + b)^2 - (a - b)^2 = 4ab$$
$$(7)^2 - (3)^2 = 4ab$$
$$49 - 9 = 4ab$$
$$10 = ab$$

Or $\quad ab = 10$

(ii) $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

This formula, square of a trinomial, involves three expressions, namely; $(a + b + c)$, $(a^2 + b^2 + c^2)$ and $2(ab + bc + ca)$. If the values of two of them are known, the value of the third expression can be calculated. The method is explained in the following example.

Example-1: If $a^2 + b^2 + c^2 = 43$ and $ab + bc + ca = 3$, then find the value of $a + b + c$.

Solution: We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$
$$(a + b + c)^2 = 43 + 2(3) = 43 + 6 = 49$$
$$\Rightarrow \quad a + b + c = \pm\sqrt{49} = \pm 7$$

Example-2: If $a + b + c = 6$ and $a^2 + b^2 + c^2 = 24$ then find the value *of ab+ be + ca.*

Solution: We have

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$(6)^2 = 24 + 2(ab + bc + ca)$$
$$36 = 24 + 2(ab + bc + ca)$$
$$12 = 2(ab + bc + ca)$$

Or $\quad ab + be + ca = 6$

Example-3: If $a + b + c = 7$ and $ab + bc + ca = 9$, then find the value of $a^2 + b^2 + c^2$.

Solution: We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2. + 2ab + 2bc + 2ca$$
$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$
$$(7)^2 = a^2+b^2+c^2+2(9)$$
$$49 = a^2 + b^2 + c^2 + 18$$

Or $\quad a^2 + b^2 + c^2 = 49 - 18 = 31$

(iii) $(a + b)^3 = a^3 + 3ab(a + b) + b^3$ and $(a - b)^3 = a^3 - 3ab(a - b) - b^3$

Example-1: If $2x - 3y = 10$ and $xy = 2$, then find the value of $8x^3 - 27y^3$.

Solution: Given $\quad 2x - 3y = 10$

$$(2x-3y)^3 = (10)^3$$
$$8x^3-27y^3-3\times2x\times3y\,(2x-3y) = 1000$$
$$8x^3-27y^3-18xy\,(2x-3y) = 1000$$
$$8x^3-27y^3-18\times2\times10 = 1000$$
$$8x^3-27y^3-360 = 1000$$
$$8x^3-27y^3 = 1000+360 = 1360$$

Example-2: If $x+\frac{1}{x}=8$, then find the value of $x^3+\frac{1}{x^3}$.

Solution: Given $x+\frac{1}{x}=8$

$\Rightarrow$ $\left(x+\frac{1}{x}\right)^3=(8)^3$

$\Rightarrow$ $x^3+\frac{1}{x^3}+3\times x\times\frac{1}{x}\left(x+\frac{1}{x}\right)=512$

$\Rightarrow$ $x^3+\frac{1}{x^3}+3\times\left(x+\frac{1}{x}\right)=512$

$\Rightarrow$ $x^3+\frac{1}{x^3}+3\times8=512$

$\Rightarrow$ $x^3+\frac{1}{x^3}+24=512$

$\Rightarrow$ $x^3+\frac{1}{x^3}=488$

Example-3: If $x-\frac{1}{x}=4$ find $x^3-\frac{1}{x^3}$.

Solution: We have $x-\frac{1}{x}=4$

$\Rightarrow$ $\left(x-\frac{1}{x}\right)^3=64$

$\Rightarrow$ $x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)=64$

$\Rightarrow$ $x^3-\frac{1}{x^3}-3(4)=64$

$\Rightarrow$ $x^3-\frac{1}{x^3}-12=64$

$\Rightarrow$ $x^3-\frac{1}{x^3}=64+12$

$$\Rightarrow \qquad x^3 - \frac{1}{x^3} = 76$$

(iv) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

The procedure for finding the products of $\left(x \pm \frac{1}{x}\right)$ and $x^2 + \frac{1}{x^2} \mp 1$ is explained in the following examples.

Example-1: Factorize $64x^3 + 343y^3$

Solution: $64x^3 + 343y^3 = (4x)^3 + (7y)^3$

$$= (4x + 7y)\left[(4x)^2 - (4x)(7y) + (7y)^2\right]$$

$$= (4x + 7y)(16x^2 - 28xy + 49y^2)$$

Example-2: Factorize $125x^3 - 1331y^3$

Solution: $125x^3 - 1331y^3 = (5x)^3 - (11y)^3$

$$= (5x - 11y)\left[(5x)^2 + (5x)(11y) + (11y)^2\right]$$

$$= (5x - 11y)(25x^2 + 55xy + 121y^2)$$

Example-3: Find the product $\left(\frac{2}{3}x + \frac{3}{2x}\right)\left(\frac{4}{9}x^2 - 1 + \frac{9}{4x^2}\right)$

Solution:

$$\left(\frac{2}{3}x + \frac{3}{2x}\right)\left(\frac{4}{9}x^2 - 1 + \frac{9}{4x^2}\right) = \left(\frac{2}{3}x + \frac{3}{2x}\right)\left[\left(\frac{2}{3}x\right)^2 - \left(\frac{2}{3}x\right)\left(\frac{3}{2x}\right) + \left(\frac{3}{2x}\right)^2\right]$$

$$= \left(\frac{2}{3}x\right)^3 + \left(\frac{3}{2x}\right)^3 = \frac{8}{27}x^3 + \frac{27}{8x^3}$$

Example-4: Find the product $\left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16}{25}x^2 + \frac{25}{16x^2} + 1\right)$

Solution: $\left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16}{25}x^2 + \frac{25}{16x^2} + 1\right) = \left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16x^2}{25} + 1 + \frac{25}{16x^2}\right)$

$$= \left(\frac{4}{5}x - \frac{5}{4x}\right)\left[\left(\frac{4}{5}x\right)^2 + \left(\frac{4}{5}x\right)\left(\frac{5}{4x}\right) + \left(\frac{5}{4x}\right)^2\right]$$

$$= \left(\frac{4}{5}x\right)^3 - \left(\frac{5}{4x}\right)^3 = \frac{64}{125}x^3 - \frac{125}{64x^3}$$

Example-5: Find the continued product of $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

Solution: $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2) = (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

$$= (x^3+y^3)(x^3-y^3) = (x^3)^2 - (y^3)^2 = x^6 - y^6$$

Solved Exercise 4.2

1. (i) If $a+b=10$ and $a-b=6$, then find the value of (a^2+b^2).

Solution: Given $a+b=10$, $a-b=6$, $a^2+b^2=?$

We know that

$$2(a^2+b^2)=(a+b)^2+(a-b)^2$$
$$2(a^2+b^2)=(10)^2+(6)^2=100+36=136$$
$$a^2+b^2=\frac{136}{2}=68$$

(ii) If $a+b=5$, $a-b=\sqrt{17}$, then find the value of ab.

Solution: Given $a+b=5$, $a-b=\sqrt{17}$

We know that

$$4ab=(a+b)^2-(a-b)^2$$
$$4ab=(5)^2-(\sqrt{17})^2=25-17=8$$
$$ab=\frac{8}{4}=2$$

2. If $a^2+b^2+c^2=45$ and $a+b+c=-1$, then find the value of $ab+bc+ca$.

Solution: Given $a^2+b^2+c^2=45$, $a+b+c=-1$, $ab+bc+ca=?$

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$$
$$(-1)^2=45+2(ab+bc+ca)$$
$$1=45+2(ab+bc+ca)$$
$$-44=2(ab+bc+ca)$$

Or $ab+bc+ca=-22$

3. If $m+n+p=10$ and $mn+np+mp=27$, then find the value of $m^2+n^2+p^2$.

Solution: Given $m+n+p=10$, $mn+np+mp=27$, $m^2+n^2+p^2=?$

We know that

$$(m+n+p)^2=m^2+n^2+p^2+2(mn+np+mp)$$
$$(10)^2=m^2+n^2+p^2+2(27)$$
$$100=m^2+n^2+p^2+54$$
$$100-54=m^2+n^2+p^2$$
$$46=m^2+n^2+p^2$$
$$m^2+n^2+p^2=46$$

4. If $x^2+y^2+z^2=78$ and $xy+yz+zx=59$, then find the value of $x+y+z$.

Solution: Given $x^2+y^2+z^2=78$, $xy+yz+zx=59$, $x+y+z=?$

We know that

$$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)$$

$$(x+y+z)^2 = 78+2(59)$$
$$(x+y+z)^2 = 78+118$$
$$(x+y+z)^2 = 196$$
$$\sqrt{(x+y+z)^2} = +\sqrt{196}$$
$$x+y+z = \pm 14$$

5. If $x+y+z=12$ and $x^2+y^2+z^2=64$, then find the value of $xy+yz+zx$.

Solution: Given $x+y+z=12$, $x^2+y^2+z^2=64$, $xy+yz+zx=?$

We know that

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$$
$$(12)^2 = 64+2(xy+yz+zx)$$
$$144 = 64+2(xy+yz+zx)$$
$$144-64 = 2(xy+yz+zx)$$
$$80 = 2(xy+yz+zx)$$
$$\frac{80}{2} = xy+yz+zx$$
$$40 = xy+yz+zx$$
$$xy+yz+zx = 40$$

6. If $x+y=7$ and $xy=12$, then find the value of x^3+y^3.

Solution: Given $x+y=7$, $xy=12$, $x^3+y^3=?$

Now $x+y=7$

$$(x+y)^3 = (7)^3$$
$$x^3+y^3+3xy(x+y) = 343$$
$$x^3+y^3+3(12)(7) = 343$$
$$x^3+y^3+252 = 343$$
$$x^3+y^3 = 343-252 = 91$$

7. If $3x+4y=11$ and $xy=12$, then find the value of $27x^3+64y^3$.

Solution: Given $3x+4y=11$, $xy=12$, $27x^3+64y^3=?$

Now $3x+4y=11$

$$(3x+4y)^3 = (11)^3$$
$$(3x)^3+(4y)^3+3(3x)(4y)(3x+4y) = 1331$$
$$27x^3+64y^3+36(xy)(11) = 1331$$
$$27x^3+64y^3+36(12)(11) = 1331$$
$$27x^3+64y^3+4752 = 1331$$
$$27x^3+64y^3 = 1331-4752 = -3421$$

8. If $x - y = 4$ and $xy = 21$, then find the value of $x^3 - y^3$.

Solution: Given $x - y = 4$, $xy = 21$, $x^3 - y^3 = ?$

$$x - y = 4$$
$$(x-y)^3 = (4)^3$$
$$x^3 - y^3 - 3(xy)(x-y) = 64$$
$$x^3 - y^3 - 3(21)(4) = 64$$
$$x^3 - y^3 - 252 = 64$$
$$x^3 - y^3 = 64 + 252 = 316$$

9. If $5x - 6y = 13$ and $xy = 6$, then find the value of $125x^3 - 216y^3$.

Solution: Given $5x - 6y = 13$, $xy = 6$, $125x^3 - 216y^3 = ?$

$$5x - 6y = 13$$
$$(5x-6y)^3 = (13)^3$$
$$(5x)^3 - (6y)^3 - 3(5x)(6y)(5x-6y) = 2197$$
$$125x^3 - 216y^3 - 90(xy)(5x-6y) = 2197$$
$$125x^3 - 216y^3 - 90(6)(13) = 2197$$
$$125x^3 - 216y^3 - 7020 = 2197$$
$$125x^3 - 216y^3 = 2197 + 7020 = 9217$$

10. If $x + \frac{1}{x} = 3$, then find the value of $x^3 + \frac{1}{x^3}$.

Solution: Given $x + \frac{1}{x} = 3$, $x^3 + \frac{1}{x^3} = ?$

$$x + \frac{1}{x} = 3$$
$$\left(x + \frac{1}{x}\right)^3 = (3)^3$$
$$x^3 + \frac{1}{x^3} + 3(x)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right) = 27$$
$$x^3 + \frac{1}{x^3} + 3(3) = 27$$
$$x^3 + \frac{1}{x^3} + 9 = 27$$
$$x^3 + \frac{1}{x^3} = 27 - 9 = 18$$

11. If $x-\frac{1}{x}=7$, then find the value of $x^3-\frac{1}{x^3}$.

Solution: Given $x-\frac{1}{x}=7$, $x^3-\frac{1}{x^3}=?$

$$x-\frac{1}{x}=7$$

$$\left(x-\frac{1}{x}\right)^3=(7)^3$$

$$x^3-\frac{1}{x^3}-3(x)\left(\frac{1}{x}\right)\left(x-\frac{1}{x}\right)=343$$

$$x^3-\frac{1}{x^3}-3(7)=343$$

$$x^3-\frac{1}{x^3}-21=343$$

$$x^3-\frac{1}{x^3}=343+21=364$$

12. If $\left(3x+\frac{1}{3x}\right)=5$, then find the value of $\left(27x^3+\frac{1}{27x^3}\right)$.

Solution: Given $3x+\frac{1}{3x}=5$, $27x^3+\frac{1}{27x^3}=?$

$$3x+\frac{1}{3x}=5$$

$$\left(3x+\frac{1}{3x}\right)^3=(5)^3$$

$$(3x)^3+\left(\frac{1}{3x}\right)^3+3\left(3x+\frac{1}{3x}\right)=125$$

$$27x^3+\frac{1}{27x^3}+3(5)=125$$

$$27x^3+\frac{1}{27x^3}+15=125$$

$$27x^3+\frac{1}{27x^3}=125-15=110$$

13. If $\left(5x-\frac{1}{5x}\right)=6$, then find the value of $\left(125x^3-\frac{1}{125x^3}\right)$.

Solution: Given $5x-\frac{1}{5x}=6$, $125x^3-\frac{1}{125x^3}=?$

$$5x-\frac{1}{5x}=6$$

$$\left(5x-\frac{1}{5x}\right)^3=(6)^3$$

$$(5x)^3-\left(\frac{1}{5x}\right)^3-3(5x)\left(\frac{1}{5x}\right)\left(5x-\frac{1}{5x}\right)=216$$

$$125x^3-\frac{1}{125x^3}-3(6)=216$$

$$125x^3-\frac{1}{125x^3}-18=216$$

$$125x^3-\frac{1}{125x^3}=216+18=234$$

14. Factorize (i) x^3-y^3-x+y **(ii)** $8x^3-\frac{1}{27y^3}$

Solution (i): $x^3-y^3-x+y=(x^3-y^3)-(x-y)$

$$=(x-y)(x^2+xy+y^2)-(x-y)$$

$$=(x-y)\left[(x^2+xy+y^2)-1\right]$$

$$=(x-y)(x^2+xy+y^2-1)$$

Solution (ii):

$$8x^3-\frac{1}{27y^3}=(2x)^3-\left(\frac{1}{3y}\right)^3=\left(2x-\frac{1}{3y}\right)\left[(2x)^2+(2x)\left(\frac{1}{3y}\right)+\left(\frac{1}{3y}\right)^2\right]$$

$$=\left(2x-\frac{1}{3y}\right)\left(4x^2+\frac{2x}{3y}+\frac{1}{9y^2}\right)$$

15. Find the products, using formulas.

(i) $(x^2+y^2)(x^4-x^2y^2+y^4)$

Solution: $(x^2+y^2)(x^4-x^2y^2+y^4)=(x^2+y^2)\left[(x^2)^2-(x^2)(y^2)+(y^2)^2\right]$

$$=(x^2)^3+(y^2)^3=x^6+y^6$$

(ii) $(x^3-y^3)(x^6+x^3y^3+y^6)$

Solution:

$$(x^3-y^3)(x^6+x^3y^3+y^6)=(x^3-y^3)\left[(x^3)^2+(x^3)(y^3)+(y^3)^2\right]$$

$$=(x^3)^3-(y^3)^3=x^9-y^9$$

(iii) $(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$

Solution:

$$(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$$
$$=\left[(x-y)(x^2+xy+y^2)\right]\left[(x+y)(x^2-xy+y^2)\right]\left[(x^2+y^2)(x^4-x^2y^2+y^4)\right]$$
$$=(x^3-y^3)(x^3+y^3)(x^6+y^6)$$
$$=\left[(x^3)^2-(y^3)^2\right](x^6+y^6)$$
$$=(x^6-y^6)(x^6+y^6)=(x^6)^2-(y^6)^2=x^{12}-y^{12}$$

(iv) $(2x^2-1)(2x^2+1)(4x^4+2x^2+1)(4x^4-2x^2+1)$

Solution: $(2x^2-1)(2x^2+1)(4x^4+2x^2+1)(4x^4-2x^2+1)$

$$=(2x^2-1)(4x^4+2x^2+1)(2x^2+1)(4x^4-2x^2+1)$$
$$=\left[(2x^2)^3-(1)^3\right]\left[(2x^2)^3+(1)^3\right]$$
$$=(8x^6-1)(8x^6+1)$$
$$=(8x^6)^2-(1)^2=64x^{12}-1$$

SURDS AND THEIR APPLICATION

Definition:

An irrational radical with rational radicand is called a surd. **For example,**

Hence the radical $\sqrt[n]{a}$ is a surd if

(i) a is rational,

(ii) the result $\sqrt[n]{a}$ is irrational.

e.g. $\sqrt{3}, \sqrt{2/5}, \sqrt[3]{7}, \sqrt[4]{10}$ are surds.

But $\sqrt{\pi}$ and $\sqrt{2+\sqrt{17}}$ are not surds because π and $2+\sqrt{17}$ are not rational.

Remember:

Note that for the surd $\sqrt[n]{a}$ is called surd index or the order of the surd and the rational number 'a' is called the radicand.

Every surd is an irrational number but every irrational number is not a surd. **e.g.,** the surd $\sqrt[3]{5}$ is an irrational number but the irrational number $\sqrt{\pi}$ is not a surd.

Operations on surds:

(a) Addition and Subtraction of Surds:

Similar surds Can be added or subtracted into a single term is explained in the following examples.

Example: Simplify by combining similar terms.

(i) $4\sqrt{3}-3\sqrt{27}+2\sqrt{75}$ (ii) $\sqrt[3]{128}-\sqrt[3]{250}+\sqrt[3]{432}$

Solution:

(i) $4\sqrt{3}-3\sqrt{27}+2\sqrt{75}=4\sqrt{3}-3\sqrt{9\times3}+2\sqrt{25\times3}=4\sqrt{3}-3\sqrt{9}\sqrt{3}+2\sqrt{25}\times\sqrt{3}$

$$=4\sqrt{3}-9\sqrt{3}+10\sqrt{3}=(4-9+10)\sqrt{3}=5\sqrt{3}$$

(ii) $\sqrt[3]{128}-\sqrt[3]{250}+\sqrt[3]{432}=\sqrt[3]{64\times2}-\sqrt[3]{125\times2}+\sqrt[3]{216\times2}$

$$=\sqrt[3]{(4)^3\times2}-\sqrt[3]{(5)^3\times2}+\sqrt[3]{(6)^3\times2}$$

$$=\sqrt[3]{(4)^3}\times\sqrt[3]{2}-\sqrt[3]{(5)^3}\times\sqrt[3]{2}+\sqrt[3]{(6)^3}\times\sqrt[3]{2}$$

$$=4\sqrt[3]{2}-5\sqrt[3]{2}+6\sqrt[3]{2}=(4-5+6)\sqrt[3]{2}=5\sqrt[3]{2}$$

(b) Multiplication and Division of Surds:

We can multiply and divide surds of the same order by making use of the following laws of surds.

$$\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}.$$

And the result obtained will be a surd of the same order.

If surds to be multiplied or divided are not of the same order, they must be reduced to the surds of the same order.

Example: Simplify and express the answer in the simplest form.

(i) $\sqrt{14}\sqrt{35}$ (ii) $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$

Solution:

(i) $\sqrt{14}\sqrt{35}=\sqrt{14\times35}=\sqrt{7\times2\times7\times5}=\sqrt{(7)^2\times2\times5}$

$$=\sqrt{(7)^2\times10}=\sqrt{(7)^2}\times\sqrt{10}=7\sqrt{10}$$

(ii) We have $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$

For $\sqrt{3}\sqrt[3]{2}$ the L.C.M. of orders 2 and 3 is 6.

Thus $\sqrt{3}=(3)^{1/2}=(3)^{3/6}=\sqrt[6]{3^3}=\sqrt[6]{27}$

And $\sqrt[3]{2}=(2)^{1/3}=(2)^{2/6}=\sqrt[6]{(2)^2}=\sqrt[6]{4}$

Hence $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}=\dfrac{\sqrt[6]{12}}{\sqrt[6]{27}\sqrt[6]{4}}=\dfrac{\sqrt[6]{12}}{\sqrt[6]{108}}=\sqrt[6]{\dfrac{12}{108}}=\sqrt[6]{\dfrac{1}{9}}$

Its simplest form is

$$\sqrt[6]{\left(\frac{1}{3}\right)^2}=\left(\frac{1}{3}\right)^{2/6}=\left(\frac{1}{3}\right)^{1/3}=\sqrt[3]{\frac{1}{3}}$$

Solved Exercise 4.3

1. Express each of the following surd in the simplest form.

(i) $\sqrt{180}$

Solution: $\sqrt{180}=\sqrt{2\times2\times3\times3\times5}=\sqrt{2^2\times3^2\times5}=6\sqrt{5}$

(ii) $3\sqrt{162}$

Solution: $3\sqrt{162}=3\sqrt{2\times3\times3\times3\times3}=3\sqrt{2\times3^2\times3^2}$

$=3\times3\times3\sqrt{2}=27\sqrt{2}$

(iii) $\frac{3}{4}\sqrt[3]{128}$

Solution: $\frac{3}{4}\sqrt[3]{128}=\frac{3}{4}(128)^{1/3}=\frac{3}{4}(2\times2\times2\times2\times2\times2\times2)^{\frac{1}{3}}=\frac{3}{4}\sqrt[3]{2^3\times2^3\times2}$

$=2\times2\times\frac{3}{4}\sqrt[3]{2}=3\sqrt[3]{2}=3\sqrt[3]{2}$

(iv) $\sqrt[5]{96x^6y^7z^8}$

Solution: $\sqrt[5]{96x^6y^7z^8}=\sqrt[5]{2\times2\times2\times2\times2\times3x^6y^7z^8}$

$=\left(2^5\times3\times x^6\times y^7\times z^8\right)^{\frac{1}{5}}=\left(2^{5\times\frac{1}{5}}\times3^{\frac{1}{5}}\times x^{\frac{6}{5}}\times y^{\frac{7}{5}}\times z^{\frac{8}{5}}\right)=2xyz\left(3xy^2z^3\right)^{\frac{1}{5}}$

$=2xyz\sqrt[5]{3xy^2z^3}$

2. Simplify

(i) $\frac{\sqrt{18}}{\sqrt{3}\sqrt{2}}$

Solution: $\frac{\sqrt{18}}{\sqrt{3}\sqrt{2}}=\frac{\sqrt{3\times2\times3}}{\sqrt{3}\sqrt{2}}=\frac{\sqrt{3}\sqrt{2}\sqrt{3}}{\sqrt{3}\sqrt{2}}=\sqrt{3}$

(ii) $\frac{\sqrt{21}\sqrt{9}}{\sqrt{63}}$

Solution: $\frac{\sqrt{21}\sqrt{9}}{\sqrt{63}}=\frac{\sqrt{7\times3}\sqrt{9}}{\sqrt{7\times9}}=\frac{\sqrt{7}\sqrt{3}\sqrt{9}}{\sqrt{7}\sqrt{9}}=\sqrt{3}$

(iii) $\sqrt[5]{243x^5y^{10}z^{15}}$

Solution: $\sqrt[5]{243x^5y^{10}z^{15}}=\left(243x^5y^{10}z^{15}\right)^{\frac{1}{5}}$

$=\left(3^5\right)^{\frac{1}{5}}\left(x^5\right)^{\frac{1}{5}}\left(y^{10}\right)^{\frac{1}{5}}\left(z^{15}\right)^{\frac{1}{5}}=3xy^2z^3$

(iv) $\frac{4}{5}\sqrt[3]{125}$

Solution: $\frac{4}{5}\sqrt[3]{125} = \frac{4}{5}(125)^{\frac{1}{3}} = \frac{4}{5}(5^3)^{\frac{1}{3}} = \frac{4}{5}(5) = 4$

(v) $\sqrt{21}\times\sqrt{7}\times\sqrt{3}$

Solution: $\sqrt{21}\times\sqrt{7}\times\sqrt{3} = \sqrt{7\times3}\times\sqrt{7}\times\sqrt{3} = \sqrt{7\times3\times7\times3} = \sqrt{7\times7\times3\times3}$

$= \sqrt{7^2\times3^2} = (7^2)^{\frac{1}{2}}\times(3^2)^{\frac{1}{2}} = 7\times3 = 21$

3. Simplify by combining similar terms.

(i) $\sqrt{45} - 3\sqrt{20} + 4\sqrt{5}$

Solution:

$$\sqrt{45} - 3\sqrt{20} + 4\sqrt{5} = \sqrt{3\times3\times5} - 3\sqrt{2\times2\times5} + 4\sqrt{5} = 3\sqrt{5} - 3\times2\sqrt{5} + 4\sqrt{5}$$

$$= (3-6+4)\sqrt{5} = (1)\sqrt{5} = \sqrt{5}$$

(ii) $4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300}$

Solution:

$$4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300} = 4\sqrt{2\times2\times3} + 5\sqrt{3\times3\times3} - 3\sqrt{3\times5\times5} + \sqrt{3\times2\times2\times5\times5}$$

$$= 4\times2\sqrt{3} + 5\times3\sqrt{3} - 3\times5\sqrt{3} + 2\times5\sqrt{3}$$

$$= 8\sqrt{3} + 15\sqrt{3} - 15\sqrt{3} + 10\sqrt{3} = (8+15-15+10)\sqrt{3} = 18\sqrt{3}$$

(iii) $\sqrt{3}\left(2\sqrt{3} + 3\sqrt{3}\right)$

Solution: $\sqrt{3}\left(2\sqrt{3} + 3\sqrt{3}\right) = \sqrt{3}\times\sqrt{3}(2+3) = \sqrt{3\times3}(5) = 3\times5 = 15$

(iv) $2\left(6\sqrt{5} - 3\sqrt{5}\right)$

Solution: $2\left(6\sqrt{5} - 3\sqrt{5}\right) = 2\sqrt{5}(6-3) = 2\sqrt{5}(3) = 2\times3\times\sqrt{5} = 6\sqrt{5}$

4. Simplify

(i) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

Solution: $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right) = (3)^2 - \left(\sqrt{3}\right)^2 = 9-3 = 6$

(ii) $(\sqrt{5}+\sqrt{3})^2$

Solution:

$$(\sqrt{5}+\sqrt{3})^2 = \left(\sqrt{5}\right)^2 + \left(\sqrt{3}\right)^2 + 2\left(\sqrt{5}\right)\left(\sqrt{3}\right) = 5+3+2\sqrt{5\times3}$$

$$= 8 + 2\sqrt{15}$$

(iii) $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$

Solution: $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right) = \left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2$

$= 5-3 = 2$

(iv) $\left(\sqrt{2}+\frac{1}{\sqrt{3}}\right)\left(\sqrt{2}-\frac{1}{\sqrt{3}}\right)$

Solution: $\left(\sqrt{2}+\frac{1}{\sqrt{3}}\right)\left(\sqrt{2}-\frac{1}{\sqrt{3}}\right)=\left(\sqrt{2}\right)^2-\left(\frac{1}{\sqrt{3}}\right)^2=2-\frac{1}{3}=\frac{6-1}{3}=\frac{5}{3}$

(v) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})(x+y)(x^2+y^2)$

Solution: $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})(x+y)(x^2+y^2)$

$$=\left[\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)\right](x+y)\left(x^2+y^2\right)$$

$$=\left[\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2\right](x+y)\left(x^2+y^2\right)$$

$$=(x-y)(x+y)\left(x^2+y^2\right)$$

$$=\left[(x-y)(x+y)\left(x^2+y^2\right)\right]=\left[(x)^2-(y)^2\right]\left(x^2+y^2\right)$$

$$=\left(x^2-y^2\right)\left(x^2+y^2\right)=\left(x^2\right)^2-\left(y^2\right)^2=x^4-y^4$$

RATIONALIZATION OF SURDS

(a) Definitions:

(i) A surd which contains a single term is called a monomial surd. e.g., $\sqrt{2},\sqrt{3}$ etc.

(ii) A surd which contains sum of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd.
e.g., $\sqrt{3}+\sqrt{7}$ or $\sqrt{2}+5$ or $\sqrt{11}-8$ etc.
We can extend this to the definition of a trinomial surd.

(iii) If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.

(iv) The process of multiplying a given surd by its rationalizing factor to get a rational number as product is called rationalization of the given surd.

(v) Two binomial surds of second order differing only in sign connecting their terms are called conjugate surds. Thus $(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are conjugate surds of each other.
The conjugate of $x+\sqrt{y}$ is $x-\sqrt{y}$.
The product of these conjugate surds $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b$ is a rational quantity independent of any radical.
Similarly, the product of $a+b\sqrt{m}$ and its conjugate $a-b\sqrt{m}$ has no radical.

For example,

$(3+\sqrt{5})(3-\sqrt{5})=(3)^2-(\sqrt{5})^2=9-5=4$, which is a rational number.

(b) Rationalizing a Denominator:

Keeping the above discussion in mind we observe that, in order to rationalize a denominator of the form $a+b\sqrt{x}$ (or $a-b\sqrt{x}$), we multiply both numerator and denominator by the conjugate factor $a-b\sqrt{x}$ (or $a+b\sqrt{x}$). By doing this we eliminate the radical and thus obtain a denominator free of any surd.

(c) Rationalizing Real Numbers of the Types $\frac{1}{a+b\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}}$

For the expression $\frac{1}{a+b\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations, where x, y are natural numbers and a, b are integers, are explained with the help of following examples.

Example-1: Rationalize the denominator $\frac{58}{7-2\sqrt{5}}$.

Solution: To rationalize the denominator, we multiply both the numerator and denominator by the conjugate $(7+2\sqrt{5})$ of $(7-2\sqrt{5})$, i.e.

$$\frac{58}{7-2\sqrt{5}} = \frac{58}{7-2\sqrt{5}} \times \frac{7+2\sqrt{5}}{7+2\sqrt{5}} = \frac{58(7+2\sqrt{5})}{(7)^2-(2\sqrt{5})^2} = \frac{58(7+2\sqrt{5})}{49-20}$$

$$= \frac{58(7+2\sqrt{5})}{29} = 2(7+\sqrt{5})$$

Example-2: Rationalize the denominator $\frac{2}{\sqrt{5}+\sqrt{2}}$.

Solution: Multiply both the numerator and denominator by the conjugate $(\sqrt{5}-\sqrt{2})$ of $(\sqrt{5}+\sqrt{2})$, to get

$$\frac{2}{\sqrt{5}+\sqrt{2}} = \frac{2}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{2(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{2(\sqrt{5}-\sqrt{2})}{5-2} = \frac{2(\sqrt{5}-\sqrt{2})}{3}$$

Example-3: Simplify $\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$.

Solution: First we shall rationalize the denominators and then simplify. We have

$$\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$$

$$= \frac{6}{2\sqrt{3}-\sqrt{6}} \times \frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

$$= \frac{6(2\sqrt{3}+\sqrt{6})}{(2\sqrt{3})^2-(\sqrt{6})^2} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2-(\sqrt{2})^2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2}$$

$$= \frac{6(2\sqrt{3}+\sqrt{6})}{12-6} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{6-2}$$

$$=\frac{12\sqrt{3}+6\sqrt{6}}{6}+\frac{\sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2}}{1}-\frac{4\sqrt{3}\sqrt{6}+4\sqrt{3}\sqrt{2}}{4}$$
$$=2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6}=0$$

Example-4: Find rational numbers x and y such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=x+y\sqrt{5}$.

Solution:

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=\frac{4+3\sqrt{5}}{4-3\sqrt{5}}\times\frac{4+3\sqrt{5}}{4+3\sqrt{5}}=\frac{(4+3\sqrt{5})^2}{(4)^2-(3\sqrt{5})^2}=\frac{16+24\sqrt{5}+45}{16-45}=\frac{61+24\sqrt{5}}{-29}$$

$$\frac{-61}{29}-\frac{24}{29}\sqrt{5}=x+y\sqrt{5} \qquad \text{(given)}$$

Hence, on comparing the two sides, we get

$$x=-\frac{61}{29}, \qquad y=-\frac{24}{29}$$

Example-5: If $x=3+\sqrt{8}$, then evaluate (i) $x+\frac{1}{x}$ (ii) $x^2+\frac{1}{x^2}$.

Solution: Since $x=3+\sqrt{8}$

$$\frac{1}{x}=\frac{1}{3+\sqrt{8}}=\frac{1}{3+\sqrt{8}}\times\frac{3-\sqrt{8}}{3-\sqrt{8}}=\frac{3-\sqrt{8}}{(3)^2-(\sqrt{8})^2}=\frac{3-\sqrt{8}}{9-8}=3-\sqrt{8}$$

(i) $$x+\frac{1}{x}=3+\sqrt{8}+3-\sqrt{8}=6$$

(ii) $$\left(x+\frac{1}{x}\right)^2=(6)^2$$

$$x^2+2(x)\left(\frac{1}{x}\right)+\frac{1}{x^2}=36$$

$$x^2+\frac{1}{x^2}=36-2=34$$

Solved Exercise 4.4

1. Rationalize the denominator of the following.

(i) $\frac{3}{4\sqrt{3}}$

Solution: $$\frac{3}{4\sqrt{3}}=\frac{3}{4\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{3\sqrt{3}}{4\left(\sqrt{3}\right)^2}=\frac{3\sqrt{3}}{4\times3}=\frac{\sqrt{3}}{4}$$

(ii) $\frac{14}{\sqrt{98}}$

Solution: $$\frac{14}{\sqrt{98}}=\frac{14}{\sqrt{98}}\times\frac{\sqrt{98}}{\sqrt{98}}=\frac{14\sqrt{98}}{\left(\sqrt{98}\right)^2}=\frac{14\sqrt{98}}{98}=\frac{\sqrt{98}}{7}$$

$$=\frac{\sqrt{7\times7\times2}}{7}=\frac{7\sqrt{2}}{7}=\sqrt{2}$$

(iii) $\frac{6}{\sqrt{8}\sqrt{27}}$

Solution: $\frac{6}{\sqrt{8}\sqrt{27}}=\frac{6}{\sqrt{216}}\times\frac{\sqrt{216}}{\sqrt{216}}=\frac{6\sqrt{216}}{\left(\sqrt{216}\right)^2}$

$$=\frac{6\sqrt{6\times6\times6}}{216}=\frac{6\times6\sqrt{6}}{216}=\frac{\sqrt{6}}{6}$$

(iv) $\frac{1}{3+2\sqrt{5}}$

Solution: $\frac{1}{3+2\sqrt{5}}=\frac{1}{3+2\sqrt{5}}\times\frac{3-2\sqrt{5}}{3-2\sqrt{5}}=\frac{3-2\sqrt{5}}{(3)^2-\left(2\sqrt{5}\right)^2}$

$$=\frac{3-2\sqrt{5}}{9-4\times5}=\frac{3-2\sqrt{5}}{9-20}=\frac{3-2\sqrt{5}}{-11}=-\frac{1}{11}\left(3-2\sqrt{5}\right)$$

(v) $\frac{15}{\sqrt{31}-4}$

Solution: $\frac{15}{\sqrt{31}-4}=\frac{15}{\sqrt{31}-4}\times\frac{\sqrt{31}+4}{\sqrt{31}+4}=\frac{15\left(\sqrt{31}+4\right)}{\left(\sqrt{31}\right)^2-(4)^2}$

$$=\frac{15\left(\sqrt{31}+4\right)}{31-16}=\frac{15\left(\sqrt{31}+4\right)}{15}=\sqrt{31}+4$$

(vi) $\frac{2}{\sqrt{5}-\sqrt{3}}$

Solution: $\frac{2}{\sqrt{5}-\sqrt{3}}=\frac{2}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{2\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$

$$=\frac{2\left(\sqrt{5}+\sqrt{3}\right)}{5-3}=\frac{2\left(\sqrt{5}+\sqrt{3}\right)}{2}=\sqrt{5}+\sqrt{3}$$

(vii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

Solution: $\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}=\frac{\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}\right)^2-(1)^2}$

$$=\frac{\left(\sqrt{3}\right)^2+(1)^2-2\left(\sqrt{3}\right)(1)}{3-1}=\frac{3+1-2\sqrt{3}}{2}$$

$$=\frac{4-2\sqrt{3}}{2}=\frac{2\left(2-\sqrt{3}\right)}{2}=2-\sqrt{3}$$

(viii) $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

Solution: $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{\left(\sqrt{5}+\sqrt{3}\right)^2}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$

$=\frac{\left(\sqrt{5}\right)^2+\left(\sqrt{3}\right)^2+2\left(\sqrt{5}\right)\left(\sqrt{3}\right)}{5-3}=\frac{5+3+2\sqrt{15}}{2}$

$=\frac{8+2\sqrt{15}}{2}=\frac{2\left(4+\sqrt{15}\right)}{2}=4+\sqrt{15}$

2. Find the conjugate of $x+\sqrt{y}$.

(i) $3+\sqrt{7}$

Solution: Suppose $z=3+\sqrt{7}$

The conjugate of z is $\bar{z}=\overline{3+\sqrt{7}}=3-\sqrt{7}$

(ii) $4-\sqrt{5}$

Solution: Suppose $z=4-\sqrt{5}$

The conjugate of z is $\bar{z}=\overline{4-\sqrt{5}}=4+\sqrt{5}$

(iii) $2+\sqrt{3}$

Solution: Suppose $z=2+\sqrt{3}$

The conjugate of z is $\bar{z}=\overline{2+\sqrt{3}}=2-\sqrt{3}$

(iv) $2+\sqrt{5}$

Solution: Suppose $z=2+\sqrt{5}$

The conjugate of z is $\bar{z}=\overline{2+\sqrt{5}}=2-\sqrt{5}$

(v) $5+\sqrt{7}$

Solution: Suppose $z=5+\sqrt{7}$

The conjugate of z is $\bar{z}=\overline{5+\sqrt{7}}=5-\sqrt{7}$

(vi) $4-\sqrt{15}$

Solution: Suppose $z=4-\sqrt{15}$

The conjugate of z is $\bar{z}=\overline{4-\sqrt{15}}=4+\sqrt{15}$

(vii) $7-\sqrt{6}$

Solution: Suppose $z=7-\sqrt{6}$

The conjugate of z is $\bar{z}=\overline{7-\sqrt{6}}=7+\sqrt{6}$

(viii) $9+\sqrt{2}$

Solution: Suppose $z=9+\sqrt{2}$

The conjugate of z is $\bar{z}=\overline{9+\sqrt{2}}=9-\sqrt{2}$

3. (i) If $x=2-\sqrt{3}$, find $\frac{1}{x}$.

Solution: $x=2-\sqrt{3}$

$$\frac{1}{x}=\frac{1}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{2+\sqrt{3}}{(2)^2-(\sqrt{3})^2}=\frac{2+\sqrt{3}}{4-3}=\frac{2+\sqrt{3}}{1}=2+\sqrt{3}$$

(ii) If $x=4-\sqrt{17}$, find $\frac{1}{x}$

Solution: $x=4-\sqrt{17}$

$$\frac{1}{x}=\frac{1}{4-\sqrt{17}}\times\frac{4+\sqrt{17}}{4+\sqrt{17}}=\frac{4+\sqrt{17}}{(4-\sqrt{17})(4+\sqrt{17})}=\frac{4+\sqrt{17}}{(4)^2-(\sqrt{17})^2}=\frac{4+\sqrt{17}}{16-17}=\frac{4+\sqrt{17}}{-1}$$

$$=-(4+\sqrt{17})=-4-\sqrt{17}$$

iii) If $x=\sqrt{3}+2$ find $x+\frac{1}{x}$.

Solution: $x=\sqrt{3}+2$

$$\frac{1}{x}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}=\frac{2-\sqrt{3}}{4-3}=\frac{2-\sqrt{3}}{1}=2-\sqrt{3}$$

$$x+\frac{1}{x}=\sqrt{3}+2+2-\sqrt{3}=4$$

4. Simplify

(i) $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

Solution: $$\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}=\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{(1+\sqrt{2})(\sqrt{5}-\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}+\frac{(1-\sqrt{2})(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}$$

$$=\frac{(1+\sqrt{2})(\sqrt{5}-\sqrt{3})}{5-3}+\frac{(1-\sqrt{2})(\sqrt{5}+\sqrt{3})}{5-3}$$

$$=\frac{(1+\sqrt{2})(\sqrt{5}-\sqrt{3})}{2}+\frac{(1-\sqrt{2})(\sqrt{5}+\sqrt{3})}{2}$$

$$=\frac{(1+\sqrt{2})(\sqrt{5}-\sqrt{3})+(1-\sqrt{2})(\sqrt{5}+\sqrt{3})}{2}$$

$$=\frac{\left[(1)(\sqrt{5})-(1)(\sqrt{3})+(\sqrt{2})(\sqrt{5})-(\sqrt{2})(\sqrt{3})\right]+\left[(1)\sqrt{5}+(1)(\sqrt{3})-(\sqrt{2})(\sqrt{5})-(\sqrt{2})(\sqrt{3})\right]}{2}$$

$$=\frac{\sqrt{5}-\sqrt{3}+\sqrt{10}-\sqrt{6}+\sqrt{5}+\sqrt{3}-\sqrt{10}-\sqrt{6}}{2}$$

$$=\frac{2\sqrt{5}-2\sqrt{6}}{2}=\frac{2(\sqrt{5}-\sqrt{6})}{2}=\sqrt{5}-\sqrt{6}$$

(ii) $\frac{1}{2+\sqrt{3}}+\frac{2}{\sqrt{5}-\sqrt{3}}+\frac{1}{2+\sqrt{5}}$

Solution: $\frac{1}{2+\sqrt{3}}+\frac{2}{\sqrt{5}-\sqrt{3}}+\frac{1}{2+\sqrt{5}}$

$$=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}+\frac{2}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}+\frac{1}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}$$

$$=\frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}+\frac{2(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}+\frac{2-\sqrt{5}}{(2)^2-(\sqrt{5})^2}$$

$$=\frac{2-\sqrt{3}}{4-3}+\frac{2(\sqrt{5}+\sqrt{3})}{5-3}+\frac{2-\sqrt{5}}{4-5}$$

$$=\frac{2-\sqrt{3}}{1}+\frac{2(\sqrt{5}+\sqrt{3})}{2}+\frac{2-\sqrt{5}}{4-5}$$

$$=(2-\sqrt{3})+(\sqrt{5}+\sqrt{3})+\frac{(2-\sqrt{5})}{-1}=(2-\sqrt{3})+(\sqrt{5}+\sqrt{3})-(2-\sqrt{5})$$

$$=2-\sqrt{3}+\sqrt{5}+\sqrt{3}-2+\sqrt{5}\quad=2\sqrt{5}$$

(iii) $\frac{2}{\sqrt{5}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{2}}-\frac{3}{\sqrt{5}+\sqrt{2}}$

Solution: $\frac{2}{\sqrt{5}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{2}}-\frac{3}{\sqrt{5}+\sqrt{2}}$

$$=\frac{2}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{3}{\sqrt{5}+\sqrt{2}}\times\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$=\frac{2(\sqrt{5}-\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}+\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}-\frac{3(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2}$$

$$=\frac{2(\sqrt{5}-\sqrt{3})}{5-3}+\frac{\sqrt{3}-\sqrt{2}}{3-2}-\frac{3(\sqrt{5}-\sqrt{2})}{5-2}$$

$$=\frac{2(\sqrt{5}-\sqrt{3})}{2}+\frac{\sqrt{3}-\sqrt{2}}{1}-\frac{3(\sqrt{5}-\sqrt{2})}{3}$$

$$=\sqrt{5}-\sqrt{3}+\sqrt{3}-\sqrt{2}-\sqrt{5}+\sqrt{2}$$

$$=0$$

5. (i) If $x = 2+\sqrt{3}$, find the value of $x - \frac{1}{x}$ and $\left(x-\frac{1}{x}\right)^2$.

Solution: Since $x = 2+\sqrt{3}$

$$\frac{1}{x} = \frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

$$x - \frac{1}{x} = \left(2+\sqrt{3}\right) - \left(2-\sqrt{3}\right) = 2+\sqrt{3}-2+\sqrt{3} = 2\sqrt{3}$$

$$\left(x-\frac{1}{x}\right)^2 = \left(2\sqrt{3}\right)^2$$

$$\left(x-\frac{1}{x}\right)^2 = 12$$

(ii) If $x = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$, find the value of $x+\frac{1}{x}, x^2+\frac{1}{x^2}$ and $x^3+\frac{1}{x^3}$.

[Hint: $a^2 + b^2 = (a + b)^2 - 2ab$ and $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$]

Solution: $x = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ and $\frac{1}{x} = \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

$$x+\frac{1}{x} = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} + \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{\left(\sqrt{5}-\sqrt{2}\right)^2+\left(\sqrt{5}+\sqrt{2}\right)^2}{\left(\sqrt{5}+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{2}\right)}$$

$$= \frac{5+2-2\sqrt{10}+5+2+2\sqrt{10}}{\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2} = \frac{14}{3}$$

Now $x+\frac{1}{x} = \frac{14}{3}$

$$\left(x+\frac{1}{x}\right)^2 = \left(\frac{14}{3}\right)^2$$

$$x^2+\frac{1}{x^2}+2 = \frac{196}{9}$$

$$x^2+\frac{1}{x^2} = \frac{196}{9} - 2 = \frac{178}{9}$$

Now $x+\frac{1}{x} = \frac{14}{3}$

$$\left(x+\frac{1}{x}\right)^3 = \left(\frac{14}{3}\right)^3$$

$$x^3+\frac{1}{x^3}+3(x)\left(\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=\frac{2744}{27}$$

$$x^3+\frac{1}{x^3}+3\left(\frac{14}{3}\right)=\frac{2744}{27}$$

$$x^3+\frac{1}{x^3}=\frac{2744}{27}-14=\frac{2366}{27}$$

6. Determine the rational numbers a and b if $\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}=a+b\sqrt{3}$.

Solution: $$\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}=a+b\sqrt{3}$$

$$\frac{\left(\sqrt{3}+1\right)^2+\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}=a+b\sqrt{3}$$

$$\frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{\left(\sqrt{3}\right)^2-(1)^2}=a+b\sqrt{3}$$

$$\frac{8}{2}=a+b\sqrt{3}$$

$$4=a+b\sqrt{3}$$

Hence, on comparing the two sides, we get

$a=4,\quad b=0$

Solved Review Exercise 4

1. Multiple Choice Questions. Choose the correct answer.

(i) $4x+3y-2$ is an algebraic ....

(a) expression (b) sentence (c) equation (d) inequation

(ii) The degree of polynomial $4x^4+2x^2y$ is ....

(a) 1 (b) 2 (c) 3 (d) 4

(iii) a^3+b^3 is equal to ....

(a) $(a-b)(a^2+ab+b^2)$ (b) $(a+b)(a^2-ab+b^2)$

(c) $(a-b)(a^2-ab+b^2)$ (d) $(a-b)(a^2+ab-b^2)$

(iv) $(3+\sqrt{2})(3-\sqrt{2})$ is equal to ....

(a) 7 (b) -7 (c) -1 (d) 1

(v) Conjugate of surd $a+\sqrt{b}$ is ....

(a) $-a+\sqrt{b}$ (b) $a-\sqrt{b}$ (c) $\sqrt{a}+\sqrt{b}$ (d) $\sqrt{a}-\sqrt{b}$

(vi) $\frac{1}{a-b}-\frac{1}{a+b}$ is equal to ....

(a) $\frac{2a}{a^2-b^2}$ (b) $\frac{2b}{a^2-b^2}$ (c) $\frac{-2a}{a^2-b^2}$ (d) $\frac{-2b}{a^2-b^2}$

(vii) $\frac{a^2-b^2}{a+b}$ **is equal to ....**

(a) $(a-b)^2$ (b) $(a+b)^2$ (c) $a+b$ (d) $a-b$

(viii) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ **is equal to ....**

(a) a^2+b^2 (b) a^2-b^2 (c) $a-b$ (d) $a+b$

Ans: (i) a (ii) d (iii) b (iv) a (v) b
(vi) b (vii) d (viii) c

2. Fill in the blanks.

(i) The degree of the polynomial $x^2y^2+3xy+y^3$ is ________.

(ii) $x^2-4=$ ________

(iii) $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)$ ________

(iv) $2(a^2+b^2)=(a+b)^2+($ ________ $)^2$

(v) $\left(x-\frac{1}{x}\right)^2=$ ________

(vi) Order of surd $\sqrt[3]{x}$ is ________

(vii) $\frac{1}{2-\sqrt{3}}=$ ________

Ans: (i) 4 (ii) $(x-2)(x+2)$ (iii) $x^2-1+\frac{1}{x^2}$
(iv) $a-b$ (v) $x^2+\frac{1}{x^2}-2$ (vi) 3 (vii) $2+\sqrt{3}$

3. If $x+\frac{1}{x}=3$, find (i) $x^2+\frac{1}{x^2}$ (ii) $x^4+\frac{1}{x^4}$

Solution (i):

$$x+\frac{1}{x}=3$$

$$\left(x+\frac{1}{x}\right)^2=(3)^2$$

$$x^2+\frac{1}{x^2}+2(x)\left(\frac{1}{x}\right)=9$$

$$x^2+\frac{1}{x^2}+2=9$$

$$x^2+\frac{1}{x^2}=9-2=7$$

Solution (ii):

$$x^2+\frac{1}{x^2}=7$$

Taking square on both sides, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=(7)^2$$

$$\left(x^2\right)^2+\left(\frac{1}{x^2}\right)^2+2\left(x^2\right)\left(\frac{1}{x^2}\right)=49$$

$$x^4+\frac{1}{x^4}+2=49$$

$$x^4+\frac{1}{x^4}=49-2=47$$

4. If $x-\frac{1}{x}=2$, find

(i) $x^2+\frac{1}{x^2}$ **(ii)** $x^4+\frac{1}{x^4}$.

Solution (i) $x-\frac{1}{x}=2$

Taking square on both sides, we get

$$\left(x-\frac{1}{x}\right)^2=(2)^2$$

$$x^2+\frac{1}{x^2}-2=4$$

$$x^2+\frac{1}{x^2}=4+2=6$$

Solution (ii):

$$x^2+\frac{1}{x^2}=6$$

Taking square on both sides, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=(6)^2$$

$$\left(x^2\right)^2+\left(\frac{1}{x^2}\right)^2+2\left(x^2\right)\left(\frac{1}{x^2}\right)=36$$

$$x^4+\frac{1}{x^4}+2=36 \quad \Rightarrow \quad x^4+\frac{1}{x^4}=36-2=34$$

5. Find the value of x^3+y^3 and xy if $x+y=5$ and $x-y=3$.

Solution: Given $x+y=5$, $x-y=3$, $xy=?$

We know that

$$4xy=(x+y)^2-(x-y)^2$$
$$4xy=(5)^2-(3)^2=25-9=16$$
$$xy=\frac{16}{4}=4$$
$$(x+y)^3=x^3+y^3+3xy(x+y)$$
$$(5)^3=x^3+y^3+3(4)(5)$$
$$\Rightarrow 125-60=x^3+y^3$$
$$\Rightarrow x^3+y^3=65$$

6. If $p=2+\sqrt{3}$, find

(i) $p+\frac{1}{p}$ (ii) $p-\frac{1}{p}$ (iii) $p^2+\frac{1}{p^2}$ (iv) $p^2-\frac{1}{p^2}$

Solution: (i) $p+\frac{1}{p}$

$$p=2+\sqrt{3}$$
$$\frac{1}{p}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}$$
$$=\frac{2-\sqrt{3}}{4-3}=\frac{2-\sqrt{3}}{1}=2-\sqrt{3}$$
$$p+\frac{1}{p}=(2+\sqrt{3})+(2-\sqrt{3})=2+\sqrt{3}+2-\sqrt{3}=4$$

(ii) $p-\frac{1}{p}$

Solution: $p=2+\sqrt{3}$

$$\frac{1}{p}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}$$
$$=\frac{2-\sqrt{3}}{4-3}=\frac{2-\sqrt{3}}{1}=2-\sqrt{3}$$
$$p-\frac{1}{p}=(2+\sqrt{3})-(2-\sqrt{3})=2+\sqrt{3}-2+\sqrt{3}$$
$$=2\sqrt{3}$$

(iii) $p^2+\frac{1}{p^2}$

Solution: $p+\frac{1}{p}=4$

Taking square on both sides, we get

$$\left(p+\frac{1}{p}\right)^2=(4)^2$$

$$p^2+\frac{1}{p^2}+2(p)\left(\frac{1}{p}\right)=16$$

$$p^2+\frac{1}{p^2}+2=16$$

$$p^2+\frac{1}{p^2}=16-2=14$$

(iv) $p^2-\frac{1}{p^2}$

Solution: $p^2-\frac{1}{p^2}=\left(p+\frac{1}{p}\right)\left(p+\frac{1}{p}\right)=(4)\left(2\sqrt{3}\right)=8\sqrt{3}$

7. If $q=\sqrt{5}+2$, find

(i) $q+\frac{1}{q}$ (ii) $q-\frac{1}{q}$ (iii) $q^2+\frac{1}{q^2}$ (iv) $q^2-\frac{1}{q^2}$

Solution: (i) $q+\frac{1}{q}$

$q=2+\sqrt{5}$

$$\frac{1}{q}=\frac{1}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}=\frac{2-\sqrt{5}}{(2)^2-\left(\sqrt{5}\right)^2}=\frac{2-\sqrt{5}}{4-5}=\frac{2-\sqrt{5}}{-1}=-\left(2-\sqrt{5}\right)=-2+\sqrt{5}$$

$$q+\frac{1}{q}=2+\sqrt{5}+\left(-2+\sqrt{5}\right)=2+\sqrt{5}-2+\sqrt{5}=2\sqrt{5}$$

(ii) $q-\frac{1}{q}$

Solution:

$q=2+\sqrt{5}$

$$\frac{1}{q}=\frac{1}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}=\frac{2-\sqrt{5}}{(2)^2-\left(\sqrt{5}\right)^2}=\frac{2-\sqrt{5}}{4-5}=\frac{2-\sqrt{5}}{-1}=-\left(2-\sqrt{5}\right)=-2+\sqrt{5}$$

$$q-\frac{1}{q}=\left(2+\sqrt{5}\right)-\left(-2+\sqrt{5}\right)$$

$= 2 + \sqrt{5} + 2 - \sqrt{5} = 4$

(iii) $q^2 + \frac{1}{q^2}$

Solution: $q + \frac{1}{q} = 2\sqrt{5}$

Taking square on both sides, we get

$$\left(q + \frac{1}{q}\right)^2 = \left(2\sqrt{5}\right)^2$$

$$q^2 + \frac{1}{q^2} + 2(q)\left(\frac{1}{q}\right) = 4 \times 5$$

$$q^2 + \frac{1}{q^2} + 2 = 20$$

$$q^2 + \frac{1}{q^2} = 20 - 2 = 18$$

(iv) $q^2 - \frac{1}{q^2}$

Solution: $q + \frac{1}{q} = 2\sqrt{5}$, $q - \frac{1}{q} = 4$

$$q^2 - \frac{1}{q^2} = \left(q + \frac{1}{q}\right)\left(q - \frac{1}{q}\right)$$

Substituting the values $q + \frac{1}{q} = 2\sqrt{5}$ and $q - \frac{1}{q} = 4$, we get

$= \left(2\sqrt{5}\right)(4) = 8\sqrt{5}$

8. Simplify

(i) $\frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} - \sqrt{a^2-2}}$

Solution: $\frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} - \sqrt{a^2-2}} = \frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} - \sqrt{a^2-2}} \times \frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} + \sqrt{a^2-2}}$

$$= \frac{\left(\sqrt{a^2+2} + \sqrt{a^2-2}\right)^2}{\left(\sqrt{a^2+2}\right)^2 - \left(\sqrt{a^2-2}\right)^2}$$

$$= \frac{\left(\sqrt{a^2+2}\right)^2 + \left(\sqrt{a^2-2}\right)^2 + 2\left(\sqrt{a^2+2}\right)\sqrt{a^2-2}}{\left(a^2+2\right) - \left(a^2-2\right)}$$

$$=\frac{a^2+2+a^2-2+2\sqrt{a^4-4}}{a^2+2-a^2+2}$$

$$=\frac{2a^2+2\sqrt{a^4-4}}{4}$$

$$=\frac{2\left(a^2+\sqrt{a^4-4}\right)}{4}=\frac{a^2+\sqrt{a^4-4}}{2}$$

(ii) $\frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}}$

Solution: $\frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}}$

$$=\frac{\left(a+\sqrt{a^2-x^2}\right)-\left(a-\sqrt{a^2-x^2}\right)}{\left(a-\sqrt{a^2-x^2}\right)\left(a+\sqrt{a^2-x^2}\right)}$$

$$=\frac{a+\sqrt{a^2-x^2}-a+\sqrt{a^2-x^2}}{a^2-\left(\sqrt{a^2-x^2}\right)^2}$$

$$=\frac{2\sqrt{a^2-x^2}}{a^2-\left(a^2-x^2\right)}=\frac{2\sqrt{a^2-x^2}}{a^2-a^2+x^2}=\frac{2\sqrt{a^2-x^2}}{x^2}$$

SUMMARY

- An algebraic expression is that in which constants or variables or both are combined by basic operations.
- Polynomial means an expression with many terms.
- Degree of polynomial means highest power of variable.
- Expression in the form $\frac{p(x)}{q(x)},(q(x)\neq 0)$ is called rational expression.
- An irrational radical with rational radicand is called a surd.
- In $\sqrt[n]{x}$, n is called surd index or surd order and rational number x is called radicand.
- A surd which contains a single term is called monomial surd.
- A surd which contains sum or difference of two surds is called binomial surd.
- Conjugate surd of $\sqrt{x}+\sqrt{y}$ is defined as $\sqrt{x}-\sqrt{y}$

UNIT 5

FACTORIZATION

Unit Outlines

5.1 *Factorization*
5.2 *Remainder Theorem and Factor Theorem*
5.3 *Factorization of a Cubic Polynomial*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- recall factorization of expressions of the following types
  - ★ $ka + kb + kc$ ★ $ac + ad + bc + bd$ ★ $a^2 \pm 2ab + b^2$
  - ★ $a^2 - b^2$ ★ $a^2 \pm 2ab + b^2 - c^2$
- factorize the expressions of the following types

Type I: $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

Type II: $x^2 + px + q$

Type III: $ax^2 + bx + c$

Type IV: $\begin{cases} (ax^2 + bx + c)(ax^2 + bx + d) + k \\ (x + a)(x + b)(x + c)(x + d) + k \\ (x + a)(x + b)(x + c)(x + d) + kx^2 \end{cases}$

Type V: $a^3 + 3a^2b + 3ab^2 + b^3$
$a^3 - 3a^2b + 3ab^2 - b^3$

Type VI: $a^3 \pm b^3$

- state and prove Remainder theorem and explain through examples
- find Remainder (without dividing) when a polynomial is divided by a linear polynomial
- define zeros of a polynomial
- state and prove Factor theorem
- use Factor theorem to factorize a cubic polynomial.

Introduction:

Factorization plays an important role in mathematics as it helps to reduce the study of a complicated expression to the study of simpler expressions. In this unit we will deal with different types of factorization of polynomials.

Factorization:

If a polynomial $p(x)$ can be expressed as $p(x) = g(x).h(x)$, then each of the polynomials $g(x)$ and $h(x)$ is called a factor of $p(x)$. For instance, in the distributive property

$ab + ac = a(b + c)$

a and (b + c) are factors of (ab + ac)

When a polynomial has been written as a product consisting only of prime factors, then it is said to be factored completely.

(a) Factorization of the Expression of the type ka + kb + kc:

Example-1: Factorize 5a – 5b + 5c

Solution: $5a - 5b + 5c = 5(a - b + c)$

Example-2: Factorize 5a – 5b – 15c

Solution: $5a - 5b - 15c = 5(a - b - 3c)$

(b) Factorization of the Expression of the type ac + ad + bc + bd:

$$\begin{aligned} ac + ad + bc + bd &= (ac + ad) + (bc + db) \\ &= a(c + d) + b(c + d) \\ &= (a + b)(c + d) \end{aligned}$$

Example-1: Factorize 3x – 3 a + xy – ay.

Solution: $\begin{aligned} 3x + xy - 3a - ay &= x(3 + y) - a(3 + y) \\ &= (3 + y)(x - a) \end{aligned}$

Example-2: Factorize $pqr + qr^2 - pr^2 - r^3$.

Solution: $\begin{aligned} pqr + qr^2 - pr^2 - r^3 &= r(pq + qr - pr - r^2) \\ &= r[(pq + qr) - pr - r^2] \\ &= r[q(p + r) - r(p + r)] \\ &= r(p + r)(q - r) \end{aligned}$

(c) Factorization of the Expression of the type $a^2 \pm 2ab + b^2$:

We know that

(i) $a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b)$

(ii) $a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b)$

Example-1: Factorize $25x^2 + 16 + 40x$.

Solution: $\begin{aligned} 25x^2 + 40x + 16 &= (5x)^2 + 2(5x)(4) + (4)^2 \\ &= (5x + 4)^2 \\ &= (5x + 4)(5x + 4) \end{aligned}$

Example-2: Factorize $12x^2 - 36x + 27$.

Solution: $\begin{aligned} 12x^2 - 36x + 27 &= 3(4x^2 - 12x + 9) \\ &= 3(2x - 3)^2 \\ &= 3(2x - 3)(2x - 3) \end{aligned}$

(d) Factorization of the Expression of the type $a^2 - b^2$:

Example: Factorize: (i) $4x^2 - (2y - z)^2$ (ii) $6x^4 - 96$

Solution: (i) $\begin{aligned} 4x^2 - (2y - z)^2 &= (2x)^2 - (2y - z)^2 \\ &= [2x - (2y - z)][2x + (2y - z)] \\ &= (2x - 2y + z)(2x + 2y - z) \end{aligned}$

(ii) $\begin{aligned} 6x^4 - 96 &= 6(x^4 - 16) \\ &= 6[(x^2)^2 - (4)^2] \end{aligned}$

$= 6(x^2 - 4)(x^2 + 4)$
$= 6[(x)^2 - (2)^2](x^2 + 4)$
$= 6(x - 2)(x + 2)(x^2 + 4)$

(e) Factorization of the Expression of the types $a^2 \pm 2ab + b^2 - c^2$:

We know that

$$a^2 \pm 2ab + b^2 - c^2 = (a \pm b)^2 - (c)^2 = (a \pm b - c)(a \pm b + c)$$

Example: Factorize (i) $x^2 + 6x + 9 - 4y^2$ (ii) $1 + 2ab - a^2 - b^2$

Solution: (i) $x^2 + 6x + 9 - 4y^2 = (x+3)^2 - (2y)^2$
$= (x + 3 + 2y)(x + 3 - 2y)$

(ii) $1 + 2ab - a^2 - b^2 = 1 - (a^2 - 2ab + b^2)$
$= (1)^2 - (a - b)^2$
$= [1 - (a - b)][1 + (a - b)]$
$= (1 - a + b)(1 + a - b)$

Solved Exercise 5.1

Factorize

1. (i) $2abc - 4abx + 2abd$

Solution: $2abc - 4abx + 2abd = 2ab(c - 2x + d)$

(ii) $9xy - 12x^2y + 18y^2$

Solution: $9xy - 12x^2y + 18y^2 = 3y(3x - 4x^2 + 6y)$

(iii) $-3x^2y - 3x + 9xy^2$

Solution: $-3x^2y - 3x + 9xy^2 = -3x(xy + 1 - 3y^2)$

(iv) $5ab^2c^3 - 10a^2b^3c - 20a^3bc^2$

Solution: $5ab^2c^3 - 10a^2b^3c - 20a^3bc^2 = 5abc(bc^2 - 2ab^2 - 4a^2c)$

(v) $3x^3y(x - 3y) - 7x^2y^2(x - 3y)$

Solution: $3x^3y(x - 3y) - 7x^2y^2(x - 3y) = (3x^3y - 7x^2y^2)(x - 3y)$
$= x^2y(3x - 7y)(x - 3y)$

(vi) $2xy^3(x^2 + 5) + 8xy^2(x^2 + 5)$

Solution: $2xy^3(x^2 + 5) + 8xy^2(x^2+5) = (2xy^3 + 8xy^2)(x^2 + 5) = 2xy^2(y + 4)(x^2 + 5)$

2. (i) $5ax - 3ay - 5bx + 3by$

Solution: $5ax - 3ay - 5bx + 3by = a(5x - 3y) - b(5x - 3y) = (a - b)(5x - 3y)$

(ii) $3xy + 2y - 12x - 8$

Solution: $3xy + 2y - 12x - 8 = y(3x + 2) - 4(3x + 2) = (y - 4)(3x + 2)$

(iii) $x^3 + 3xy^2 - 2x^2y - 6y^3$

Solution: $x^3 + 3xy^2 - 2x^2y - 6y^3 = x(x^2 + 3y^2) - 2y(x^2 + 3y^2) = (x - 2y)(x^2 + 3y^2)$

(iv) $(x^2 - y^2)z + (y^2 - z^2)x$

Solution: $(x^2 - y^2)z + (y^2 - z^2)x = x^2z - y^2z + xy^2 - xz^2 = x^2z - xz^2 + xy^2 - y^2z$
$= xz(x - z) + y^2(x - z) = (x - z)(xz + y^2)$

3. (i) $144a^2 + 24a + 1$

Solution: $144a^2 + 24a + 1 = (12a)^2 + 2(12a)(1) + (1)^2 = (12a + 1)^2$

(ii) $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$

Solution: $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2} = \left(\frac{a}{b}\right)^2 - 2\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2 = \left(\frac{a}{b} - \frac{b}{a}\right)^2$

(iii) $(x+y)^2 - 14z(x+y) + 49z^2$

Solution: $(x+y)^2 - 14z(x+y) + 49z^2 = (x+y)^2 - 14z(x+y) + 49z^2$

$= (x+y)^2 - 2(x+y)(7z) + (7z)^2$

$= (x+y-7z)^2$

(iv) $12x^2 - 36x + 27$

Solution: $12x^2 - 36x + 27 = 3(4x^2 - 12x + 9) = 3\left[(2x)^2 - 2(2x)(3) + (3)^2\right]$

$= 3(2x-3)^2$

4. (i) $3x^2 - 75y^2$

Solution: $3x^2 - 75y^2 = 3(x^2 - 25y^2) = 3\left[(x)^2 - (5y)^2\right] = 3(x-5y)(x+5y)$

(ii) $x(x-1) - y(y-1)$

Solution: $x(x-1) - y(y-1) = x^2 - x - y^2 + y = x^2 - y^2 - x + y$

$= (x^2 - y^2) - (x - y) = (x-y)(x+y) - (x-y)$

$= (x-y)(x+y-1)$

(iii) $128am^2 - 242an^2$

Solution: $128am^2 - 242an^2 = 2a(64m^2 - 121n^2) = 2a\left[(8m)^2 - (11n)^2\right]$

$= 2a(8m - 11n)(8m + 11n)$

(iv) $3x - 243x^3$

Solution: $3x - 243x^3 = 3x(1 - 81x^2) = 3x\left[(1)^2 - (9x)^2\right]$

$= 3x(1-9x)(1+9x)$

5. (i) $x^2 - y^2 - 6y - 9$

Solution: $x^2 - y^2 - 6y - 9 = x^2 - \left[y^2 + 6y + 9\right] = x^2 - \left[(y)^2 + 2(y)(3) + (3)^2\right]$

$= x^2 - (y+3)^2 = \left[x - (y+3)\right]\left[x + (y+3)\right]$

$= (x-y-3)(x+y+3)$

(ii) $x^2 - a^2 + 2a - 1$

Solution: $x^2 - a^2 + 2a - 1 = x^2 - \left[a^2 - 2a + 1\right] = x^2 - \left[(a)^2 - 2(a)(1) + (1)^2\right]$

$= x^2 - (a-1)^2 = \left[x - a + 1\right]\left[x + (a-1)\right]$

$= (x-a+1)(x+a-1)$

(iii) $4x^2 - y^2 - 2y - 1$

Solution: $4x^2 - y^2 - 2y - 1 = 4x^2 - (y^2 + 2y + 1) = (2x)^2 - (y+1)^2$

$$=\left[2x-(y+1)\right]\left[2x+(y+1)\right]=(2x-y-1)(2x+y+1)$$

(iv) $x^2-y^2-4x-2y+3$

Solution: $x^2-y^2-4x-2y+3 = x^2-4x+4-y^2-2y-1$

$$=\left[(x)^2-2(x)(2)+(2)^2\right]-\left(y^2+2y+1\right)$$
$$=(x-2)^2-\left[(y)^2+2(y)(1)+(1)^2\right]=(x-2)^2-(y+1)^2$$
$$=\left[(x-2)-(y+1)\right]\left[(x-2)+(y+1)\right]$$
$$=(x-2-y-1)(x-2+y+1)$$
$$=(x-y-3)(x+y-1)$$

(v) $25x^2-10x+1-36z^2$

Solution: $25x^2-10x+1-36z^2=\left[(5x)^2-2(5x)(1)+(1)^2\right]-(6z)^2=(5x-1)^2-(6z)^2$

$$=(5x-1-6z)(5x-1+6z)$$

(vi) $x^2-y^2-4xz+4z^2$

Solution: $x^2-y^2-4xz+4z^2=\left[x^2-4xz+4z^2\right]-y^2$

$$=\left[(x)^2-2(x)(2z)+(2z)^2-(y)^2\right]=(x-2z)^2-(y)^2$$
$$=(x-y-2z)(x+y-2z)$$

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(a) Factorization of the Expression of types

$a^4+a^2b^2+b^4$ or a^4+4b^4:

Example-1: Factorize $81x^4+36x^2y^2+16y^4$.

Solution: $81x^4+36x^2y^2+16y^4=(9x^2)^2+72x^2y^2+(4y^2)^2-36x^2y^2$

$$=(9x^2+4y^2)^2-(6xy)^2$$
$$=(9x^2+4y^2+6xy)(9x^2+4y^2-6xy)$$
$$=(9x^2+6xy+4y^2)(9x^2-6xy+4y^2)$$

Example-2: Factorize $9x^4+36y^4$

Solution: $9x^4+36y^4 = 9x^4+36x^2y^2+36x\,y^4-36x^2y^2$

$$=(3x^2)^2+2(3x^2)(6y^2)+(6y^2)^2-(6xy)^2$$
$$=(3x^2+6y^2)^2-(6xy)^2$$
$$=(3x^2+6y^2+6xy)(3x^2+6y^2-6xy)$$
$$=(3x^2+6xy+6y^2)(3x^2-6xy+6y^2)$$

(b) Factorization of the Expression of the type x^2+px+q:

Example: Factorize (i) $x^2-7x+12$ (ii) $x^2+5x-36$

Solution: (i) $x^2-7x+12=x^2-3x-4x+12$

$$=x(x-3)-4(x-3)$$
$$=(x-3)(x-4)$$

(ii) $x^2 + 5x - 36 = x^2 + 9x - 4x - 36$
$= x(x + 9) - 4(x + 9)$
$= (x + 9)(x - 4)$

(c) Factorization of the Expression of the type $ax^2 + bx + c,\ a \neq 0$:

Example: Factorize: (i) $9x^2 + 21x - 8$ (ii) $2x^2 - 8x - 42$ (iii) $10x^2 - 41xy + 21y^2$

Solution:

(i) $9x^2 + 21x - 8 = 9x^2 + 24x - 3x - 8$
$= 3x(3x + 8) - 1(3x + 8)$
$= (3x + 8)(3x - 1)$

(ii) $2x^2 - 8x - 42 = 2(x^2 - 4x - 21)$
$= 2[x^2 + 3x - 7x - 21]$
$= 2[x(x + 3) - 7(x + 3)]$
$= 2[(x + 3)(x - 7)]$

(iii) $10x^2 - 41xy + 21y^2 = 10x^2 - 35xy - 6xy + 21y^2$
$= 5x(2x - 7y) - 3y(2x - 7y)$
$= (2x - 7y)(5x - 3y)$

(d) Factorization of the following types of Expressions:

$(ax^2 + bx + c)(ax^2 + bx + d) + k$
$(x + a)(x + b)(x + c)(x + d) + k$
$(x + a)(x + b)(x + c)(x + d) + kx^2$

Example-1: Factorize $(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Solution: Suppose that $y = x^2 - 4x$

$(y - 5)(y - 12) - 144 = y^2 - 17y - 84$
$= y^2 - 21y + 4y - 84$
$= y(y - 21) + 4(y - 21)$
$= (y - 21)(y + 4)$
$= (x^2 - 4x - 21)(x^2 - 4x + 4)$ (since $y = x^2 - 4x$)
$= (x^2 - 7x + 3x - 21)(x - 2)^2$
$= [x(x - 7) + 3(x - 7)](x - 2)^2$
$= (x - 7)(x + 3)(x - 2)(x - 2)$

Example-2: Factorize $(x + 1)(x + 2)(x + 3)(x + 4) - 120$

Solution: We observe that $1 + 4 = 2 + 3$.
It suggests that we rewrite the given expression as:
$[(x + 1)(x + 4)][(x + 2)(x + 3)] - 120$
$(x^2 + 5x + 4)(x^2 + 5x + 6) - 120$

Let $x^2 + 5x = y$, then, we get
$(y + 4)(y + 6) - 120 = y^2 + 10y + 24 - 120$
$= y^2 + 10y - 96$
$= y^2 + 16y - 6y - 96$
$= y(y + 16) - 6(y + 16)$

$= (y + 16)(y - 6)$
$= (x^2 + 5x + 16)(x^2 + 5x - 6)$ since $y = x^2 + 5x$
$= (x^2 + 5x + 16)(x + 6)(x - 1)$

Example-3: Factorize $(x^2 - 5x + 6)(x^2 - 5x + 6) - 2x^2$

Solution: $(x^2 - 5x + 6)(x^2 + 5x + 6) - 2x^2$
$= [x^2 - 3x - 2x + 6][x^2 + 3x + 2x + 6] - 2x^2$
$= [x(x - 3) - 2(x - 3)][x(x + 3) + 2(x + 3)] - 2x^2$
$= [(x - 3)(x - 2)][(x + 3)(x + 2)] - 2x^2$
$= [(x - 2)(x + 2)][(x - 3)(x + 3)] - 2x^2$
$= (x^2 - 4)(x^2 - 9) - 2x^2$
$= x^4 - 13x^2 + 36 - 2x^2$
$= x^4 - 15x^2 + 36$
$= x^4 - 12x^2 - 3x^2 + 36$
$= x^2(x^2 - 12) - 3(x^2 - 12)$
$= (x^2 - 12)(x^2 - 3)$
$= [(x)^2 - (2\sqrt{3})^2][(x)^2 - (\sqrt{3})^2]$
$= (x - 2\sqrt{3})(x + 2\sqrt{3})(x - \sqrt{3})(x + \sqrt{3})$

(e) Factorization of Expressions of the following Types:

$a^3 + 3a^2b + 3ab^2 + b^3$
$a^3 - 3a^2b + 3ab^2 - b^3$

Example: Factorize $x^3 - 8y^3 - 6x^2y + 12xy^2$

Solution: $x^3 - 8y^3 - 6x^2y + 12xy^2$
$= (x)^3 - (2y)^3 - 3(x)^2(2y) + 3(x)(2y)^2$
$= (x)^3 - 3(x)^2(2y) + 3(x)(2y)^2 - (2y)^3$
$= (x - 2y)^3$
$= (x - 2y)(x - 2y)(x - 2y)$

(d) Factorization of Expressions of the Following types $a^3 \pm b^3$:

We recall the formulas,
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example-1: Factorize $27x^3 + 64y^3$

Solution: $27x^3 + 64y^3 = (3x)^3 + (4y)^3$
$= (3x + 4y)[(3x)^2 - (3x)(4y) + (4y)^2]$
$= (3x + 4y)(9x^2 - 12xy + 16y^2)$

Example-2: Factorize $1 - 125x^3$

Solution: $1 - 125x^3 = (1)^3 - (5x)^3$
$= (1 - 5x)[(1)^2 + (1)(5x) + (5x)^2]$
$= (1 - 5x)(1 + 5x + 25x^2)$

Solved Exercise 5.2

Factorize

1. (i) $x^4+\frac{1}{x^4}-3$

Solution: $x^4+\frac{1}{x^4}-3=x^4+\frac{1}{x^4}-2-1$

$$=x^4-2+\frac{1}{x^4}-1$$

$$=\left[\left(x^2\right)^2-2\left(x^2\right)\left(\frac{1}{x^2}\right)+\left(\frac{1}{x^2}\right)^2\right]-1$$

$$=\left(x^2-\frac{1}{x^2}\right)^2-(1)^2$$

$$=\left(x^2-\frac{1}{x^2}-1\right)\left(x^2-\frac{1}{x^2}+1\right)$$

(ii) $3x^4+12y^4$

Solution: $3x^4+12y^4=3\left(x^4+4y^4\right)$

$$=3\left[x^4+4y^4+4x^2y^2-4x^2y^2\right]$$

$$=3\left[\left\{\left(x^2\right)^2+\left(2y^2\right)^2+2\left(x^2\right)\left(2y^2\right)\right\}-4x^2y^2\right]$$

$$=3\left[\left(x^2+2y^2\right)^2-(2xy)^2\right]$$

$$=3\left(x^2+2y^2-2xy\right)\left(x^2+2y^2+2xy\right)$$

(iii) $a^4+3a^2b^2+4b^4$

Solution: $a^4+3a^2b^2+4b^4=a^4+4a^2b^2+4b^4-a^2b^2$

$$=\left[\left(a^2\right)^2+2\left(a^2\right)\left(2b^2\right)+\left(2b^2\right)^2\right]-(ab)^2$$

$$=\left(a^2+2b^2\right)^2-(ab)^2=\left(a^2+2b^2+ab\right)\left(a^2+2b^2-ab\right)$$

(iv) $4x^4+81$

Solution: $4x^4+81=4x^4+81=4x^4+81+36x^2-36x^2$

$$=\left[(2x^2)^2+(9)^2+2\left(2x^2\right)(9)\right]-\left(36x^2\right)$$

$$=\left(2x^2+9\right)^2-(6x)^2$$

$$=\left(2x^2+9+6x\right)\left(2x^2+9-6x\right)$$
$$=\left(2x^2+6x+9\right)\left(2x^2-6x+9\right)$$

(v) x^4+x^2+25

Solution: $x^4+x^2+25=x^4+25+x^2$

$$=\left[\left(x^2\right)^2+(5)^2+2\left(x^2\right)(5)\right]-10x^2+x^2$$
$$=\left(x^2+5\right)^2-9x^2=\left(x^2+5\right)^2-(3x)^2$$
$$=\left(x^2+5+3x\right)\left(x^2+5-3x\right)=\left(x^2+3x+5\right)\left(x^2-3x+5\right)$$

(vi) x^4+4x^2+16

Solution: $x^4+4x^2+16 \quad =x^4+16+4x^2$

$$=\left[\left(x^2\right)^2+(4)^2+2\left(x^2\right)(4)\right]-8x^2+4x^2$$
$$=\left(x^2+4\right)^2-4x^2$$
$$=\left(x^2+4\right)^2-(2x)^2$$
$$=\left(x^2+4+2x\right)\left(x^2+4-2x\right)$$
$$=\left(x^2+2x+4\right)\left(x^2-2x+4\right)$$

2. (i) $x^2+14x+48$

Solution: $x^2+14x+48=x^2+8x+6x+48$

$$=x(x+8)+6(x+8)$$
$$=(x+8)(x+6)$$

(ii) $x^2-21x+108$

Solution: $x^2-21x+108=x^2-12x-9x+108$

$$=x(x-12)-9(x-12)$$
$$=(x-12)(x-9)$$

(iii) $x^2-11x-42$

Solution: $x^2-11x-42=x^2-14x+3x-42$

$$=x(x-14)+3(x-14)$$
$$=(x-14)(x+3)$$

(iv) $x^2+x-132$

Solution: $x^2+x-132=x^2+12x-11x-132$

$$=x(x+12)-11(x+12)$$
$$=(x-11)(x+12)$$

3. (i) $4x^2+12x+5$

Solution: $4x^2+12x+5 \quad = 4x^2+10x+2x+5$

$= 2x(2x+5)+1(2x+5) = (2x+5)(2x+1)$

(ii) $30x^2+7x-15$

Solution: $30x^2+7x-15 \quad = 30x^2+25x-18x-15$

$= 5x(6x+5)-3(6x+5) = (6x+5)(5x-3)$

(iii) $24x^2-65x+21$

Solution: $24x^2-65x+21 = 24x^2-56x-9x+21$

$= 8x(3x-7)-3(3x-7) = (8x-3)(3x-7)$

(iv) $5x^2-16x-21$

Solution: $5x^2-16x-21 = 5x^2-21x+5x-21$

$= x(5x-21)+1(5x-21) = (5x-21)(x+1)$

(v) $4x^2-17xy+4y^2$

Solution: $4x^2-17xy+4y^2 = 4x^2-16xy-xy+4y^2$

$= 4x(x-4y)-y(x-4y) = (4x-y)(x-4y)$

(vi) $3x^2-38xy-13y^2$

Solution: $3x^2-38xy-13y^2 = 3x^2-39xy+xy-13y^2$

$= 3x(x-13y)+y(x-13y) = (x-13y)(3x+y)$

(vii) $5x^2+33xy-14y^2$

Solution: $5x^2+33xy-14y^2 = 5x^2+35xy-2xy-14y^2$

$= 5x(x+7y)-2y(x+7y) = (5x-2y)(x+7y)$

(viii) $\left(5x-\frac{1}{x}\right)^2+4\left(5x-\frac{1}{x}\right)+4, x\neq 0$

Solution: $\left(5x-\frac{1}{x}\right)^2+4\left(5x-\frac{1}{x}\right)+4 = \left(5x-\frac{1}{x}\right)^2+2\left(5x-\frac{1}{x}\right)(2)+(2)^2$

$= \left(5x-\frac{1}{x}+2\right)^2 = \left(5x-\frac{1}{x}+2\right)\left(5x-\frac{1}{x}+2\right)$

4. (i) $(x^2+5x+4)(x^2+5x+6)-3$

Solution: $(x^2+5x+4)(x^2-5x+6)-3$

Let $x^2+5x=y$, then we get

$= (y+4)(y+6)-3$

$= y^2+6y+4y+24-3$

$= y^2+10y+21$

$= y^2 + 7y + 3y + 21$
$= y(y+7) + 3(y+7)$
$= (y+3)(y+7)$
$= (x^2 + 5x + 3)(x^2 + 5x + 7)$ Since $y = x^2 + 5x$

(ii) $(x^2 - 4x)(x^2 - 4x - 1) - 20$

Solution: $(x^2 - 4x)(x^2 - 4x - 1) - 20$

Let $x^2 - 4x = y$, then we get

$= (y)(y-1) - 20$
$= y^2 - y - 20$
$= y^2 - 5y + 4y - 20$
$= y(y-5) + 4(y-5)$
$= (y-5)(y+4)$
$= (x^2 - 4x - 5)(x^2 - 4x + 4)$ since $y = x^2 - 4x$
$= (x^2 - 5x + x - 5)(x^2 - 2x - 2x + 4)$
$= [x(x-5) + 1(x-5)][x(x-2) - 2(x-2)]$
$= (x-5)(x+1)(x-2)(x-2)$
$= (x-5)(x+1)(x-2)^2$

(iii) $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

Solution: $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

$= [(x+2)(x+5)][(x+3)(x+4)] - 15$
$= (x^2 + 5x + 2x + 10)(x^2 + 4x + 3x + 12) - 15$
$= (x^2 + 7x + 10)(x^2 + 7x + 12) - 15$

Let $x^2 + 7x = y$, then we get

$= (y+10)(y+12) - 15 = y^2 + 12y + 10y + 120 - 15$
$= y^2 + 22y + 120 - 15$
$= y^2 + 22y + 105$
$= y^2 + 15y + 7y + 105$
$= y(y+15) + 7(y+15)$
$= (y+15)(y+7)$
$= (x^2 + 7x + 15)(x^2 + 7x + 7)$ Since $y = x^2 + 7x$

(iv) $(x+4)(x-5)(x+6)(x-7)-504$

Solution: $(x+4)(x-5)(x+6)(x-7)-504$

$$=\left[(x+4)(x-5)\right]\left[(x+6)(x-7)\right]-504$$
$$=\left(x^2-5x+4x-20\right)\left(x^2-7x+6x-42\right)-504$$
$$=\left(x^2-x-20\right)\left(x^2-x-42\right)-504$$

Let $x^2-x=y$, then we get

$$=(y-20)(y-42)-504$$
$$=y^2-42y-20y+840-504$$
$$=y^2-62y+336$$
$$=y^2-56y-6y+336$$
$$=y(y-56)-6(y-56)$$
$$=(y-56)(y-6)$$
$$=\left(x^2-x-56\right)\left(x^2-x-6\right) \quad \text{Since } y=x^2-x$$
$$=\left[x^2-8x+7x-56\right]\left[x^2-3x+2x-6\right]$$
$$=\left[x(x-8)+7(x-8)\right]\left[x(x-3)+2(x-3)\right]$$
$$=(x-8)(x+7)(x-3)(x+2)$$

(v) $(x+1)(x+2)(x+3)(x+6)-3x^2$

Solution: $(x+1)(x+2)(x+3)(x+6)-3x^2$

$$=\left[(x+1)(x+6)\right]\left[(x+2)(x+3)\right]-3x^2$$
$$=\left(x^2+6x+x+6\right)\left(x^2+2x+3x+6\right)-3x^2$$
$$=\left(x^2+7x+6\right)\left(x^2+5x+6\right)-3x^2$$
$$=x\left(x+7+\frac{6}{x}\right)x\left(x+5+\frac{6}{x}\right)-3x^2$$
$$=x^2\left(x+\frac{6}{x}+7\right)\left(x+\frac{6}{x}+5\right)-3x^2$$

Let $y=x+\dfrac{6}{x}$

$$= x^2(y+7)(y+5)-3x^2$$
$$= x^2[(y+7)(y+5)-3]$$
$$= x^2[y^2+5y+7y+35-3]$$
$$= x^2[y^2+12y+32]$$
$$= x^2[y^2+8y+4y+32]$$
$$= x^2[y(y+8)+4(y+8)]$$

$= x^2(y+4)(y+8)$ Since $y = x + \frac{6}{x}$

$= x^2\left[\left(x+\frac{6}{x}+4\right)\left(x+\frac{6}{x}+8\right)\right]$

5. (i) $x^3 + 48x - 12x^2 - 64$

Solution: $x^3 + 48x - 12x^2 - 64 = (x)^3 - 3(x)^2(4) + 3(x)(4)^2 - (4)^3$

$= (x-4)^3$

(ii) $8x^3 + 60x^2 + 150x + 125$

Solution: $8x^3 + 60x^2 + 150x + 125 = (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$

$= (2x+5)^3$

(iii) $x^3 - 18x^2 + 108x - 216$

Solution: $x^3 - 18x^2 + 108x - 216 = (x)^3 - 3(x)^2(6) + 3(x)(6)^2 - (6)^3$

$= (x-6)^3$

(iv) $8x^3 - 125y^3 - 60x^2y + 150xy^2$

Solution: $8x^3 - 125y^3 - 60x^2y + 150xy^2 = (2x)^3 - (5y)^3 - 3(2x)^2(5y) + 3(2x)(5y)^2$

$= (2x-5y)^3$

6. (i) $27 + 8x^3$

Solution: $27 + 8x^3 = (3)^3 + (2x)^3$

$= (3+2x)\left[(3)^2 - (3)(2x) + (2x)^2\right]$

$= (3+2x)(9-6x+4x^2)$

(ii) $125x^3 - 216y^3$

Solution: $125x^3 - 216y^3 = (5x)^3 - (6y)^3$

$= (5x-6y)\left[(5x)^2 + (5x)(6y) + (6y)^2\right]$

$= (5x-6y)(25x^2 + 30xy + 36y^2)$

(iii) $64x^3 + 27y^3$

Solution: $64x^3 + 27y^3 = (4x)^3 + (3y)^3$

$= (4x+3y)\left[(4x)^2 - (4x)(3y) + (3y)^2\right]$

$= (4x+3y)(16x^2 - 12xy + 9y^2)$

(iv) $8x^3 + 125y^3$

Solution:

$8x^3 + 125y^3 = (2x)^3 + (5y)^3$

$= (2x+5y)\left[(2x)^2 - (2x)(5y) + (5y)^2\right]$

$= (2x+5y)(4x^2 - 10xy + 25y^2)$

REMAINDER THEOREM AND FACTOR THEOREM

Remainder Theorem:

If a polynomial p(x) is divided by a linear divisor (x – a), then the remainder is p(a).

Proof:

Let q(x) be the quotient obtained after dividing p(x) by (x - a). But the divisor (x-a) is linear. So the remainder must be of degree zero i.e., a non-zero constant, say R. Consequently, by division Algorithm we may write

$$p(x) = (x - a)\,q(x) + R$$

This is an identity in x and so is true for all real numbers x. In particular, it is true for x = a. Therefore,

$$p(a) = (a - a)\,q(a) + R = 0 + R = R$$

i.e., p(a) = the remainder.

Hence the theorem is proved.

Note: Similarly, if the divisor is (ax – b), we have

$$p(x) = (ax - b)\,q(x) + R$$

Substituting $x = \frac{b}{a}$ so that ax – b = 0, we obtain

$$p\left(\frac{b}{a}\right) = 0.\,q(x) + R = 0 + R = R$$

Thus if the divisor is linear, the above theorem provides an efficient way of finding the remainder without being involved in the process of long division.

To find Remainder (without dividing) when a polynomial is divided by a Linear Polynomial:

Example-1: Find the remainder when $9x^2 - 6x + 2$ is divided by

(i) x – 3 (ii) x + 3 (iii) 3x + 1 (iv) x

Solution: Let $p(x) = 9x^2 - 6x + 2$

(i) When p(x) is divided by x - 3, by Remainder Theorem, the remainder is
$R = p(3) = 9(3)^2 - 6(3) + 2 = 65$

(ii) When p(x) is divided by x + 3 = x – (–3), the remainder is
$$R = p(-3) = 9(-3)^2 - 6(-3) + 2 = 101$$

(iii) When p(x) is divided by 3x + 1, the remainder is
$$R = p\left(-\frac{1}{3}\right) = 9\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 2 = 5$$

(iv) When p(x) is divided by x, the remainder is
$$R = p(0) = 9(0)^2 - 6(0) + 2 = 2$$

Example-2: Find the value of k if the expression $x^3 + kx^2 + 3x - 4$ leaves a remainder of -2 when divided by $x + 2$.

Solution: Let $p(x) = x^3 + kx^2 + 3x - 4$

By the Remainder Theorem, when $p(x)$ is divided by $x + 2 = x - (-2)$, the remainder is

$$\begin{aligned} p(-2) &= (-2)^3 + k(-2)^2 + 3(-2) - 4 \\ &= -8 + 4k - 6 - 4 \\ &= 4k - 18 \end{aligned}$$

By the given condition, we have

$$p(-2) = -2 \quad \Rightarrow \quad 4k - 18 = -2 \quad \Rightarrow \quad k = 4$$

Zero of a Polynomial:

Definition:

If a specific number $x = a$ is substituted for the variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.

A very useful consequence of the remainder theorem is what is known as the factor theorem.

Factor Theorem:

The polynomial $(x - a)$ is a factor of the polynomial $p(x)$ if and only if $p(a)=0$.

Proof:

Let $q(x)$ be the quotient and R the remainder when a polynomial $p(x)$ is divided by $(x - a)$. Then by division Algorithm,

$$p(x) = (x - a)q(x)+R$$

By the Remainder Theorem, $R = p(a)$

Hence $p(x) = (x - a)\, q(x) + p(a)$

(i) Now if $p(a) = 0$, then $p(x) = (x - a)\, q(x)$
i.e., $(x - a)$ is a factor of $p(x)$

(ii) Conversely, if $(x - a)$ is a factor of $p(x)$, then the remainder upon dividing $p(x)$ by $(x - a)$ must be zero i.e., $p(a) = 0$
This completes the proof.

Note: The Factor Theorem can also be stated as, "$(x - a)$ is a factor of $p(x)$ if and only if $x = a$ is a solution of the equation $p(x) = 0$".

The Factor Theorem helps us to find factors of polynomials because it determines whether a given linear polynomial $(x - a)$ is a factor of $p(x)$. All we need is to check whether $p(a) = 0$.

Example-1: Determine if $(x - 2)$ is a factor of $x^3 - 4x^2 + 3x + 2$.

Solution: Let $p(x) = x^3 - 4x^2 + 3x + 2$

Then the remainder for $(x - 2)$ is

$$\begin{aligned} p(2) &= (2)^3 - 4(2)^2 + 3(2) + 2 \\ &= 8 - 16 + 6 + 2 = 0 \end{aligned}$$

Hence by Factor Theorem, $(x - 2)$ is a factor of the polynomial $p(x)$.

Example-2: Find a polynomial p(x) of degree 3 that has 2, –1, and 3 as zeros (i.e., roots).

Solution: Since $x = 2, -1, 3$ are roots of $p(x) = 0$

So, by Factor Theorem $(x - 2)$, $(x + 1)$ and $(x - 3)$ are the factors of $p(x)$.

Thus $p(x) = a(x - 2)(x + 1)(x - 3)$

Where any non-zero value can be assigned to a, so

Taking $a = 1$, we get

$$p(x) = (x - 2)(x + 1)(x - 3)$$
$$= x^3 - 4x^2 + x + 6 \quad \text{is the required polynomial.}$$

Solved Exercise 5.3

1. Use the remainder theorem to find the remainder when

(i) $3x^3 - 10x^2 + 13x - 6$ is divided by $(x - 2)$

Solution: Let $p(x) = 3x^3 - 10x^2 + 13x - 6$

When $p(x)$ is divided by $(x-2)$, by Remainder Theorem, the remainder is

$$R = p(2) = 3(2)^3 - 10(2)^2 + 13(2) - 6$$
$$= 3(8) - 10(4) + 26 - 6 = 24 - 40 + 26 - 6 = 50 - 46 = 4$$

(ii) $4x^3 - 4x + 3$ is divided by $(2x - 1)$

Solution: Let $p(x) = 4x^3 - 4x + 3$

When $p(x)$ is divided by $2x - 1$, by Remainder Theorem, the remainder is

$$R = p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right) + 3$$
$$= 4\left(\frac{1}{8}\right) - 4\left(\frac{1}{2}\right) + 3 = \frac{1}{2} - 2 + 3 = \frac{1}{2} + 1 = \frac{1+2}{2} = \frac{3}{2}$$

(iii) $6x^4 + 2x^3 - x + 2$ is divided by $(x + 2)$

Solution: Let $p(x) = 6x^4 + 2x^3 - x + 2$

When $p(x)$ is divided by $x + 2 = x-(-2)$, by Remainder Theorem, the remainder is

$$R = p(-2) = 6(-2)^4 + 2(-2)^3 - (-2) + 2$$
$$= 6(16) + 2(-8) + 2 + 2$$
$$= 96 - 16 + 2 + 2 = 100 - 16 = 84$$

(iv) $(2x - 1)^3 + 6(3 + 4x)^2 - 10$ is divided by $(2x + 1)$

Solution: Let $p(x) = (2x - 1)^3 + 6(3 + 4x)^2 - 10$

When $p(x)$ is divided by $2x + 1 = 2x + (-1)$, by Remainder Theorem, the remainder is

$$R = p\left(\frac{-1}{2}\right) = \left(2\left(\frac{-1}{2}\right) - 1\right)^3 + 6\left(3 + 4\left(-\frac{1}{2}\right)\right)^2 - 10$$

$$=(-1-1)^3+6(3+2(-1))^2-10$$
$$=(-2)^3+6(3-2)^2-10=-8+6(1)^2-10=-8+6-10=6-18=-12$$

(v) $x^3-3x^2+4x-14$ is divided by $(x+2)$

Solution: Let $p(x)=x^3-3x^2+4x-14$

When $p(x)$ is divided by $x+2=x-(-2)$, by Remainder Theorem, the remainder is

$$R=p(-2)=(-2)^3-3(-2)^2+4(-2)-14$$
$$=-8-3(4)-8-14=-8-12-8-14=-42$$

2. (i) If $(x+2)$ is a factor of $3x^2-4kx-4k^2$, then find the value(s) of k.

Solution: Let $p(x)=3x^2-4kx-4k^2$

By the Remainder Theorem, when $P(x)$ is divided by $x+2=x-(-2)$, the remainder is

$$p(-2)=3(-2)^2-4k(-2)-4k^2$$
$$=3(4)+8k-4k^2=12+8k-4k^2$$

By the given condition, we have

$$p(-2)=0$$
$$12+8k-4k^2=0$$
$$-4k^2+8k+12=0$$
$$-4(k^2-2k-3)=0$$
$$\Rightarrow \quad k^2-2k-3=0$$
$$k^2-3k+k-3=0$$
$$k(k-3)+1(k-3)=0$$
$$\Rightarrow \quad k-3=0 \quad \text{or} \quad k+1=0$$
$$k=3 \qquad\qquad k=-1$$

(ii) If $(x-1)$ is a factor of $x^3-kx^2+11x-6$, then find the value of k.

Solution: Let $P(x)=x^3-kx^2+11x-6$

When $p(x)$ is divided by $x-1$, by the Remainder Theorem, the remainder is

$$p(1)=(1)^3-k(1)^2+11(1)-6$$
$$=1-k+11-6=6-k$$

By the given condition, we have

$$P(1)=0 \Rightarrow \quad 6-k=0$$
$$-k=-6 \quad \Rightarrow \quad k=6$$

3. Without actual long division determine whether

(i) $(x-2)$ and $(x-3)$ are factors of $p(x)=x^3-12x^2+44x-48$.

Solution: For convenience, let

$p(x)=x^3-12x^2+44x-48$

Then the remainder for $x-2$ is

$$p(2)=(2)^3-12(2)^2+44(2)-48$$
$$=8-12(4)+88-48=8-48+88-48=96-96=0$$

Hence by Factor Theorem, $(x-2)$ is a factor of polynomial $p(x)$

Again, let $p(x) = x^2 - 12x^3 + 44x - 48$

Then the remainder for $x - 3$ is

$$P(3) = (3)^3 - 12(3)^2 + 44(3) - 48$$

$$= 27 - 12(9) + 132 - 48 = 27 - 108 + 132 - 48 = 159 - 156 = 3$$

Hence by factor theorem $(x - 3)$ is not a factor of polynomial $p(x)$

(ii). $(x - 2), (x + 3)$ and $(x - 4)$ are factors of $q(x) = x^3 + 2x^2 - 5x - 6$

Solution: For convenience, let $q(x) = x^3 + 2x^2 - 5x - 6$

Then the remainder for $x - 2$ is

$q(2) = (2)^3 + 2(2)^2 - 5(2) - 6$
$= 8 + 8 - 10 - 6 = 16 - 16 = 0$

Hence by factor theorem $(x - 2)$ is a factor of polynomial $q(x)$

Again, let $q(x) = x^3 + 2x^2 - 5x - 6$.

Then the remainder for $x + 3 = x - (-3)$ is

$q(-3) = (-3)^3 + 2(-3)^2 - 5(-3) - 6$
$= -27 + 18 + 15 - 6 = -33 + 33 = 0$

Hence by factor theorem $(x + 3)$ is a factor of polynomial $q(x)$.

Now, again let $q(x) = x^3 + 2x^2 - 5x - 6$.

Then the remainder for $x - 4$ is

$q(4) = (4)^3 + 2(4)^2 - 5(4) - 6$
$= 64 + 32 - 20 - 6 = 96 - 26 = 70 \neq 0$

Hence by factor theorem $(x - 4)$ is not a factor of polynomial $q(x)$.

4. For what value of m is the polynomial $p(x) = 4x^3 - 7x^2 + 6x - 3m$ exactly divisible by $x + 2$?

Solution: Let $p(x) = 4x^3 - 7x^2 + 6x - 3m$

By the Remainder Theorem, when $p(x)$ is divided by $x + 2 = x - (-2)$, the remainder is

$$p(-2) = 4(-2)^3 - 7(-2)^2 + 6(-2) - 3m = 4(-8) - 7(4) - 12 - 3m$$

$$= -32 - 28 - 12 - 3m = -72 - 3m$$

By the given condition, we have

$$p(-2) = 0 \quad \Rightarrow \quad -72 - 3m = 0$$

$$-3m = 72 \quad \Rightarrow \quad m = \frac{72}{-3} \quad \Rightarrow \quad m = -24$$

5. Determine the value of k if $p(x) = kx^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + k$ leaves the same remainder when divided by $(x - 3)$.

Solution: Let $p(x) = kx^3 + 4x^2 + 3x - 4$

By the Remainder Theorem, when $p(x)$ is divided by $x - 3$, the remainder is

$$p(3) = k(3)^3 + 4(3)^2 + 3(3) - 4$$
$$= k(27) + 4(9) + 9 - 4$$
$$= 27k + 36 + 9 - 4 = 27k + 41$$

Let $q(x) = x^3 - 4x + k$

By the Remainder Theorem, when $q(x)$ is divided by

$x-3$, the remainder is

$$q(3)=(3)^3-4(3)+k$$
$$=27-12+k \quad =15+k$$

By the given condition, we have

$$p(3)=q(3)$$
$$27k+41=15+k$$
$$27k-k=15-41$$
$$26k=-26 \Rightarrow k=-1$$

6. The remainder after dividing the polynomial $p(x) = x^3+ax^2+7$ by $(x+1)$ is 2b. Calculate the value of a and b if this expression leaves a remainder of $(b+5)$ on being divided by $(x-2)$.

Solution: Let $p(x)=x^3+ax^2+7$ by $(x+1)$

By the Remainder Theorem, when $p(x)$ is divided by $x+1=x-(-1)$, the remainder is 2b.

$$p(-1)=2b$$
$$(-1)^3+a(-1)^2+7=2b$$
$$-1+a+7=2b$$
$$a+6=2b$$
$$a=2b-6 \quad \_\_\_\_\_\_ \text{(i)}$$

By the remainder Theorem, when $p(x)$ is divided by $x-2$, the remainder is $b+5$.

$$p(2)=b+5$$
$$(2)^3+a(2)^2+7=b+5$$
$$8+4a+7=b+5$$
$$4a+15=b+5$$
$$4a=b+5-15$$
$$4a=b-10$$
$$a=\frac{1}{4}(b-10) \quad \_\_\_\_\_\_ \text{(ii)}$$

By comparing eq. (i) and eq (ii), we get

$$2b-6=\frac{1}{4}(b-10)$$
$$4(2b-6)=b-10$$
$$8b-24=b-10$$
$$7b=14$$
$$b=\frac{14}{7}=2$$

Put $b=2$ in eq. (i), we get

$$a=2(2)-6 \quad =4-6=-2$$

$$a=-2\,,\; b=2$$

7. The polynomial $x^3 + lx^2 + mx + 24$ has a factor $(x + 4)$ and it leaves a remainder of 36 when divided by $(x - 2)$. Find the values of l and m.

Solution: Let $P(x) = x^3 + lx^2 + mx + 24$

By the Remainder Theorem, when p(x) is divided by $x+4 = x - (-4)$, the remainder is '0'.

$$p(-4)=0$$
$$(-4)^3+\ell(-4)^2+m(-4)+24=0$$
$$-64+16\ell-4m+24=0$$
$$16\ell-4m-40=0$$
$$4(4\ell-m-10)=0$$
$$\Rightarrow \quad 4\ell-m-10=0$$
$$4\ell-m=10 \qquad \text{\_\_\_\_\_\_(i)}$$

By the Remainder Theorem, when p(x) is divided by x−2, the remainder is 36.

$$p(2)=36$$
$$(2)^3+\ell(2)^2+m(2)+24=36$$
$$8+4\ell+2m+24=36$$
$$4\ell+2m+32=36$$
$$4\ell+2m=36-32$$
$$4\ell+2m=4$$
$$2(2\ell+m)=4$$
$$\Rightarrow \quad 2\ell+m=2 \qquad \text{\_\_\_\_\_\_(ii)}$$

Adding eq. (i) and eq. (ii), we get

$$4\ell-m=10$$
$$\underline{2\ell+m=2}$$
$$6\ell=12$$

Put $\ell = 2$ in eq (ii), we get $\quad \ell=2$

$$2(2)+m=2$$
$$4+m=2$$
$$m=2-4$$
$$m=-2$$
$$l=2,\; m=-2$$

8. The expression lx^3+mx^2-4 leaves remainder of −3 and 12 when divided by (x−1) and (x+2) respectively. Calculate the values of l and m.

Solution: Let $\quad P(x) = \ell x^3 + mx^2 - 4$

By the Remainder Theorem, when p(x) is divided by x−1, the remainder is −3.

$$p(1)=-3$$
$$\ell(1)^3+m(1)^2-4=-3$$
$$\ell+m-4=-3$$

$$\ell + m = 4 - 3$$

$$\ell + m = 1 \quad \_\_\_\_\_\_ \text{(i)}$$

By the Remainder Theorem, when p(x) is divided by x+2 = x −(−2), then remainder is 12.

$$p(-2) = 12$$
$$\ell(-2)^3 + m(-2)^2 - 4 = 12$$
$$\ell(-8) + m(4) - 4 = 12$$
$$-8\ell + 4m - 4 = 12$$
$$-8\ell + 4m = 12 + 4$$
$$-8\ell + 4m = 16$$
$$4(-2\ell + m) = 16$$
$$\Rightarrow \quad -2\ell + m = 4 \quad \_\_\_\_\_\_ \text{(ii)}$$

Subtract eq. (ii) from eq. (i), we get

$$\ell + m = 1$$
$$\mp 2\ell \pm m = \pm 4$$
$$3\ell = -3$$
$$\ell = -1$$

Put $\ell = -1$ in eq. (i), we get

$$-1 + m = 1$$
$$m = 1 + 1 = 2$$
$$\ell = -1, m = 2$$

9. The expression $ax^3 - 9x^2 + bx + 3a$ is exactly divisible by $x^2 - 5x + 6$. Find the values of a and b.

Solution: Let $p(x) = ax^3 - 9x^2 + bx + 3a$

As $x^2 - 5x + 6 = 0$

$$x^2 - 3x - 2x + 6 = 0$$
$$x(x-3) - 2(x-3) = 0$$
$$(x-3)(x-2) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x - 2 = 0$$

By the Remainder Theorem, when p(x) is divided by (x − 3), the remainder is

$$p(3) = 0$$
$$a(3)^3 - 9(3)^2 + b(3) + 3a = 0$$
$$27a - 81 + 3b + 3a = 0$$
$$30a + 3b = 81 \quad \_\_\_\_\_\_ \text{(i)}$$

By the Remainder Theorem, when p(x) is divided by (x − 2), the remainder is

$$p(2) = 0$$
$$a(2)^3 - 9(2)^2 + b(2) + 3a = 0$$
$$8a - 36 + 2b + 3a = 0$$

$$11a + 2b = 36 \quad \_\_\_\_\_ \text{(ii)}$$

Multiply eq. (i) by '2' and eq. (ii) by '3', then subtract eq. (ii) from eq. (i), we get

$$\begin{array}{r} 60a + 6b = 162 \\ \pm 33a \pm 6b = \pm 108 \\ \hline 27a = 54 \\ a = 2 \end{array}$$

Put b = 2 in eq. (i), we get

$$30(2) + 3b = 81$$
$$60 + 3b = 81$$
$$3b = 81 - 60$$
$$3b = 21$$
$$b = 7$$
$$a = 2, b = 7$$

Factorization of a Cubic Polynomial:

We can use Factor Theorem to factorize a cubic polynomial as explained below. This is a convenient method particularly for factorization of a cubic polynomial. We state (without proof) a very useful Theorem.

Rational Root Theorem:

Let $a_0x^n + a_1x^{n-1} + \ldots.. + a_{n-1}x + a_n = 0, a_0 \neq 0$

be a polynomial equation of degree n with integral coefficients. If p/q is a rational root (expressed in lowest terms) of the equation, then p is a factor of the constant term a_n and q is a factor of the leading coefficient a_0.

Example: Factorize the polynomial $x^3 - 4x^2 + x + 6$, by using Factor Theorem.

Solution: We have $p(x) = x^3 - 4x^2 + x + 6$.

Possible factors of the constant term p = 6 are ±1, ±2, ±3 and ±6 and of leading coefficient q = 1 are ±1. Thus the expected zeros (or roots) of p(x) = 0 are $\frac{p}{q}$ = ±1, ±2, ± 3 and ±6. If x =a is a zero of p(x), then (x – a) will be a factor.

We use the hit and trial method to find zeros of p(x). Let us try x = 1.

Now $p(1) = (1)^3 - 4(1)^2 + 1 + 6$
$= 1 - 4 + 1 + 6 = 4 \neq 0$

Hence x = –1 is not a .zero of p(x).

Again $p(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$
$= 1 - 4 - 1 + 6 = 0$

Hence x = –1 is a zero of p(x) and therefore, x – (–1) = (x + 1) is a factor of p(x).

Now $p(2) = (2)^3 - 4(2)^2 + 2 + 6$
$= 8 - 16 + 2 + 6 = 0 \Rightarrow x = 2$ is a root.

Hence (x – 2) is also a factor of p(x).

Similarly $p(3) = (3)^3 - 4(3)^2 + 3 + 6$

$= 27 - 36 + 3 + 6 = 0 \Rightarrow x = 3$ is a zero of $p(x)$.

Hence $(x - 3)$ is the third factor of $p(x)$.

Thus the factorized form of $P(x) = x^3 - 4x^2 + x + 6$ is

$(x + 1)(x - 2)(x - 3)$

Solved Exercise 5.4

Factorize each of the following cubic polynomials by factor theorem.

1. $x^3 - 2x^2 - x + 2$

Solution: We have $p(x) = x^3 - 2x^2 - x + 2$ ______ (i)

Let $x = 1$

So (i) becomes

$P(1) = (1)^3 - 2(1)^2 - (1) + 2 = 1 - 2 - 1 + 2 = 3 - 3 = 0$

Hence $x = 1$ is a zero of $p(x)$ and therefore, $(x - 1)$ is a factor of $p(x)$.

Now again let $x = 2$

So (i) becomes

$P(2) = (2)^3 - 2(2)^2 - (2) + 2$

$= 8 - 2(4) - 2 + 2 = 8 - 8 - 2 + 2 = 10 - 10 = 0$

Hence $(x - 2)$ is also a factor of $p(x)$.

Now again let $x = -1$

So (i) becomes

$p(-1) = (-1)^3 - 2(-1)^2 - (-1) + 2$

$= -1 - 2(1) + 1 + 2 = -1 - 2 + 1 + 2 = 3 - 3 = 0$

Hence $(x - (-1)) = (x + 1)$ is the third factor of $p(x)$.

Thus the factorized from of $p(x) = x^3 - 2x^2 - x + 2$ is $(x+1)(x-1)(x-2)$

2. $x^3 - x^2 - 22x + 40$

Solution: We have $p(x) = x^3 - x^2 - 22x + 40$ ______ (i)

Let $x = 4$

So (i) becomes

Now $p(4) = (4)^3 - (4)^2 - 22(4) + 40$

$= 64 - 16 - 88 + 40 = 104 - 104 = 0$

Hence $x = 4$ is a zero of $p(x)$ and therefore, $(x - 4)$ is a factor of $p(x)$

Now again let $x = 2$

So (i) becomes

$p(2) = (2)^3 - (2)^2 - 22(2) + 40$

$= 8 - 4 - 44 + 40 = 48 - 48 = 0$

Hence (x – 2) is also a factor of p(x).
Now again let x = – 5

$$p(-5) = (-5)^3 - (-5)^2 - 22(-5) + 40$$
$$= -125 - 25 + 110 + 40 = -150 + 150 = 0$$

Hence $(x-(-5)) = (x+5)$ is the third factor of p(x).
Hence the factorized form of $p(x) = x^3 - x^2 - 22x + 40$ is $(x+5)(x-2)(x-4)$

3. $x^3 - 6x^2 + 3x + 10$

Solution: We have $p(x) = x^3 - 6x^2 + 3x + 10$ ________ (i)
Let x = – 1
So (i) becomes

Now $p(-1) = (-1)^3 - 6(-1)^2 + 3(-1) + 10$
$$= -1 - 6 - 3 + 10 = -10 + 10 = 0$$

Hence $(x-(-1)) = (x+1)$ is a zero of p(x) and therefore $(x+1)$ is a factor of p(x).

Now again let x = 2
So (i) becomes

Now $p(2) = (2)^3 - 6(2)^2 + 3(2) + 10$
$$= 8 - 6(4) + 6 + 10 = 8 - 24 + 6 + 10 = 24 - 24 = 0$$

Hence $(x-2)$ is also a factor of p(x).

Now again let x = 5
So (i) becomes

Now $p(5) = (5)^3 - 6(5)^2 + 3(5) + 10$
$$= 125 - 6(25) - 15 + 10 = 125 - 150 + 15 + 10 = 150 - 150 = 0$$

Hence (x – 5) is the third factor of p(x).
Hence the factorized form of $p(x) = x^3 - 6x^2 + 3x + 10$ is
$$(x+1)(x-2)(x-5)$$

4. $x^3 + x^2 - 10x + 8$

Solution: We have $p(x) = x^3 + x^2 - 10x + 8$ ___________ (i)
Let x = 1
So (i) becomes

Now $p(1) = (1)^3 + (1)^2 - 10(1) + 8$
$$= 1 + 1 - 10 + 8 = 10 - 10 = 0$$

Hence x = 1 is a zero of p(x) and therefore (x – 1) is a factor of p(x).
Now again let x = 2
So (i) becomes

Now $p(2) = (+2)^3 + (+2)^2 - 10(+2) + 8$
$$= +8 + 4 - 20 + 8 = 20 - 20 = 0$$

Hence (x–2) is also a factor of p(x).
Now again let $x = -4$
So (i) becomes
$P(-4) = (-4)^3 + (-4)^2 - 10(-4) + 8$
$= -64 + 16 + 40 + 8$
$= 48 - 48 = 0$
Hence $(x-(-4)) = (x+4)$ is the third factor of p(x).
Hence the factorized form of $p(x) = x^3 + x^2 - 10x + 8$ is $(x-1)(x-2)(x+4)$

5. $x^3 - 2x^2 - 5x + 6$

Solution: We have $p(x) = x^3 - 2x^2 - 5x + 6$ ________ (i)
Let $x = -2$
So (i) becomes
Now $p(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6$
$= -8 - 2(4) + 10 + 6 = -8 - 8 + 10 + 6 = -16 + 16 = 0$
Hence $x = -2$ is a zero of p(x) and therefore $(x-(-2)) = (x+2)$ is a factor of p(x).
Now again let $x = 1$
So (i) becomes
$p(1) = (1)^3 - 2(1)^2 - 5(1) + 6$
$= 1 - 2 - 5 + 6 = 7 - 7 = 0$
Hence (x–1) is also a factor of p(x).
Now again let $x = 3$
So (i) becomes
$p(3) = (3)^3 - 2(3)^2 - 5(3) + 6$
$= 27 - 2(9) - 15 + 6 = 27 - 18 - 15 + 6 = 33 - 33 = 0$
Hence $(x - 3)$ is the thrid factor of p(x).
Hence the factorized form of $p(x) = x^3 - 2x^2 - 5x + 6$ is $(x+2)(x-1)(x-3)$

6. $x^3 + 5x^2 - 2x - 24$

Solution: We have $p(x) = x^3 + 5x^2 - 2x - 24$ ________ (i)
Let $x = -4$
So (i) becomes
Now $p(-4) = (-4)^3 + 5(-4)^2 - 2(-4) - 24$
$= -64 + 5(16) + 8 - 24 = -64 + 80 + 8 - 24 = 88 - 88 = 0$
Hence $x = -4$ is a zero of p(x) and therefore $(x-(-4)) = (x+4)$ is a factor of p(x).
Now again let $x = -3$
So (i) becomes
Now $p(-3) = (-3)^3 + 5(-3)^2 - 2(-3) - 24$
$= -27 + 5(9) + 6 - 24 = -27 + 45 + 6 - 24 = 51 - 51 = 0$
Hence $(x-(-3)) = (x+3)$ is also a factor of p(x)

Now $p(2) = (2)^3 + 5(2)^2 - 2(2) - 24$

$= 8 + 5(4) - 4 - 24 = 8 + 20 - 4 - 24 = 28 - 28 = 0$

Hence $(x - 2)$ is the thrid factor of $p(x)$.
Hence the factorized form of $p(x) = x^3 + 5x^2 - 2x - 24$ is $(x+4)(x+3)(x-2)$.

7. $3x^3 - x^2 - 12x + 4$

Solution: We have $p(x) = 3x^3 - x^2 - 12x + 4$ ________ (i)
Let $x = 2$
So (i) becomes
Now $p(2) = 3(2)^3 - (2)^2 - 12(2) + 4$

$= 3(8) - 4 - 24 + 4 = 24 - 4 - 24 + 4 = 28 - 28 = 0$

Hence $x = 2$ is a zero of $p(x)$ and therefore $(x - 2)$ is a factor of $p(x)$.
Now again let $x = -2$
So (i) becomes

$$p(-2) = 3(-2)^3 - (-2)^2 - 12(-2) + 4$$
$$= 3(-8) - 4 + 24 + 4 = -24 - 4 + 24 + 4 = -28 + 28 = 0$$

Hence $(x - (-2)) = (x + 2)$ also a factor of $p(x)$

Now again let $x = \frac{1}{3}$

So (i) becomes

$$p\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{3}\right) + 4$$
$$= 3\left(\frac{1}{27}\right) - \frac{1}{9} - 4 + 4 = \frac{1}{9} - \frac{1}{9} - 4 + 4 = 0$$

Hence $(3x - 1)$ is the third factor of $p(x)$.
Hence the factorized form of $p(x) = 3x^3 - x^2 - 12x + 4$ is $(x-2)(x+2)(3x-1)$

8. $2x^3 + x^2 - 2x - 1$

Solution: We have $p(x) = 2x^3 + x^2 - 2x - 1$ ________ (i)
Let $x = -1$
So (i) becomes

$$p(-1) = 2(-1)^3 + (-1)^2 - 2(-1) - 1$$
$$= 2(-1) + 1 + 2 - 1 = -2 + 1 + 2 - 1 = 3 - 3 = 0$$

Hence $x = -1$ is a zero of $p(x)$ and therefore, $(x - (-1)) = (x + 1)$ is a factor of $p(x)$.
Now again let $x = 1$
So (i) becomes
Now $p(1) = 2(1)^3 + (1)^2 - 2(1) - 1$

$= 2 + 1 - 2 - 1 = 3 - 3 = 0$

Hence $(x-1)$ is also a factor of $p(x)$.

Now again let $x=-\frac{1}{2}$

So (i) becomes

Now $p\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2-2\left(-\frac{1}{2}\right)-1$

$$=2\left(-\frac{1}{8}\right)+\frac{1}{4}+1-1=-\frac{1}{4}+\frac{1}{4}+1-1=0$$

Hence $(2x+1)$ is the third factor of $p(x)$.

Hence the factorized form of $p(x)=2x^3+x^2-2x-1$ is

$$(x-1)(x+1)(2x+1)$$

Solved Review Exercise 5

1. Multiple Choice Questions. Choose the correct answer.

(i) The factors of x^2-5x+6 are ......

(a) $x+1, x-6$ (b) $x-2, x-3$ (c) $x+6, x-1$ (d) $x+2, x+3$

(ii) Factors of $8x^3+27y^3$ are......

(a) $(2x+3y), (4x^2+9y^2)$ (b) $(2x-3y), (4x^2-9y^2)$

(c) $(2x+3y), (4x^2-6xy+9y^2)$ (d) $(2x-3y), (4x^2+6xy+9y^2)$

(iii) Factors of $3x^2-x-2$ are ......

(a) $(x+1), (3x-2)$ (b) $(x+1), (3x+2)$

(c) $(x-1), (3x-2)$ (d) $(x-1), (3x+2)$

(iv) Factors of a^4-4b^4 are ......

(a) $(a-b),(a+b),(a^2+4b^2)$ (b) $(a^2-2b^2)(a^2+2b^2)$

(c) $(a-b),(a+b),(a^2-4b^2)$ (d) $(a-2b), (a^2+2b^2)$

(v) What will be added to complete the square of $9a^2-12ab$?..................

(a) $-16b^2$ (b) $16b^2$ (c) $4b^2$ (d) $-4b^2$

(vi) Find m so that x^2+4x+m is a complete square ......

(a) 8 (b) -8 (c) 4 (d) 16

(vii) Factors of $5x^2-17xy-12y^2$ are ......

(a) $(x+4y),(5x+3y)$ (b) $(x-4y),(5x-3y)$

(c) $(x-4y),(5x+3y)$ (d) $(5x-4y),(x+3y)$

(viii) Factors of $27x^3-\frac{1}{x^3}$ are

(a) $\left(3x-\frac{1}{x}\right),\left(9x^2+3+\frac{1}{x^2}\right)$ (b) $\left(3x+\frac{1}{x}\right),\left(9x^2+3+\frac{1}{x^2}\right)$

(c) $\left(3x-\frac{1}{x}\right),\left(9x^2-3+\frac{1}{x^2}\right)$ (d) $\left(3x+\frac{1}{x}\right),\left(9x^2-3+\frac{1}{x^2}\right)$

Answers: (i) b (ii) c (iii) d (iv) b
(v) c (vi) c (vii) c (viii) a

2. Completion Items. Fill in the blanks.

(i) $x^2 + 5x + 6 =$ ________

(ii) $4a^2 - 16 =$ ________

(iii) $4a^2 + 4ab +$ (________) is a complete square

(iv) $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} =$ ________

(v) $(x+y)(x^2 - xy + y^2) =$ ________

(vi) Factored form of $x^4 - 16$ is ________

(vii) If $x - 2$ is factor of $p(x) = x^2 + 2kx + 8$, then $k =$ ________

Answers:

(i) $(x+2)(x+3)$ (ii) $4(a-2)(a+2)$ (iii) b^2

(iv) $\left(\frac{x}{y} - \frac{y}{x}\right)^2$ (v) $x^3 + y^3$ (vi) $(x-2)(x+2)(x^2+4)$

(vii) -3

3. Factorize the following.

(i) $x^2 + 8x + 16 - 4y^2$

Solution:
$$\begin{aligned} x^2 + 8x + 16 - 4y^2 &= \left[(x)^2 + 2(x)(4) + (4)^2\right] - (2y)^2 \\ &= (x+4)^2 - (2y)^2 \\ &= (x+4+2y)(x+4-2y) \\ &= (x+2y+4)(x-2y+4) \end{aligned}$$

(ii) $4x^2 - 16y^2$

Solution:
$$\begin{aligned} 4x^2 - 16y^2 &= 4\left[x^2 - 4y^2\right] \\ &= 4\left[(x)^2 - (2y)^2\right] = 4(x-2y)(x+2y) \end{aligned}$$

(iii) $9x^2 + 27x + 8$

Solution:
$$\begin{aligned} 9x^2 + 27x + 8 &= 9x^2 + 24x + 3x + 8 \\ &= 3x(3x+8) + 1(3x+8) \\ &= (3x+8)(3x+1) \end{aligned}$$

(iv) $1 - 64z^3$

Solution:
$$\begin{aligned} 1 - 64z^3 &= (1)^3 - (4z)^3 \\ &= (1-4z)\left[(1)^2 + (1)(4z) + (4z)^2\right] \\ &= (1-4z)(1+4z+16z^2) \end{aligned}$$

(v) $8x^3 - \dfrac{1}{27y^3}$

Solution: $$8x^3 - \frac{1}{27y^3} = (2x)^3 - \left(\frac{1}{3y}\right)^3$$
$$= \left(2x - \frac{1}{3y}\right)\left[(2x)^2 + (2x)\left(\frac{1}{3y}\right) + \left(\frac{1}{3y}\right)^2\right]$$
$$= \left(2x - \frac{1}{3y}\right)\left(4x^2 + \frac{2x}{3y} + \frac{1}{9y^2}\right)$$

(vi) $2y^2 + 5y - 3$

Solution: $$2y^2 + 5y - 3 = 2y^2 + 6y - y - 3$$
$$= 2y(y+3) - 1(y+3)$$
$$= (y+3)(2y-1)$$

(vii) $x^3 + x^2 - 4x - 4$

Solution: $$x^3 + x^2 - 4x - 4 = x^3 - 4x + x^2 - 4$$
$$= x(x^2-4) + 1(x^2-4)$$
$$= (x+1)(x^2-4)$$
$$= (x+1)\left[(x)^2 - (2)^2\right]$$
$$= (x+1)(x+2)(x-2)$$

(viii) $25m^2n^2 + 10mn + 1$

Solution: $$25m^2n^2 + 10mn + 1 = 25m^2n^2 + 5mn + 5mn + 1$$
$$= 5mn(5mn+1) + 1(5mn+1)$$
$$= (5mn+1)(5mn+1)$$
$$= (5mn+1)^2$$

(ix) $1 - 12pq + 36p^2q^2$

Solution:
$$1 - 12pq + 36p^2q^2 = (1)^2 - 2(1)(6pq) + (6pq)^2$$
$$= (1-6pq)^2$$

SUMMARY

- If a polynomial is expressed as a product of other polynomials, then each polynomial in the product is called a factor of the original polynomial.
- The process of expressing an algebraic expression in terms of its factors is called factorization. We learned to factorize expressions of the following types:
  - $ka + kb + kc$
  - $ac + ad + bc + bd$
  - $a^2 \pm 2ab + b^2$
  - $a^2 - b^2$
  - $(a^2 \pm 2ab+b^2)-c^2$
  - $a^4+a^2b^2 + b^4$ or $a^4 + 4b^4$
  - $x^2 + px+q$
  - $ax^2 + bx + c$
  - $(ax^2 + bx + c)(ax^2 + bx + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + kx^2$
  - $a^3 + 3a^2b + 3ab^2 + b^3$
  - $a^3 - 3a^2b + 3ab^2 - b^3$
  - $a^3 \pm b^3$
- If a polynomial $p(x)$ is divided by a linear divisor $(x - a)$, then the remainder is $p(a)$.
- If a specific number $x = a$ is substituted for the variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.
- The polynomial $(x - a)$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$. Factor theorem has been used to factorize cubic polynomials.

UNIT 6

ALGEBRAIC MANIPULATION

Unit Outlines

6.1 *Highest Common Factor and Least Common Multiple*
6.2 *Basic Operations on Algebraic Fractions*
6.3 *Square Root of Algebraic Expression*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- find Highest Common Factor and Least Common Multiple of algebraic expressions.
- use factor or division method to determine highest common factor and least common multiple.
- know the relationship between H.C.F and L.C.M.
- solve real life problems related to H.C.F and L.C.M.
- use highest common factor and least common multiple to reduce fractional expressions involving +, −, x, ÷.
- find square root of algebraic expressions by factorization and division.

Introduction:

In this unit we will first deal with finding H.C.F. and L.C.M. of algebraic expressions by factorization and long division. Then by using H.C.F. and L.C.M. we will simplify fractional expressions. Toward the end of the unit finding square root of algebraic expression by factorization and division are discussed.

Highest Common Factor (H.C.F.) and Least Common: Multiple (L.C.M.) of Algebraic Expressions:

(a) Highest Common Factor (H.C.F.):

If two or more algebraic expressions are given then their common factor of highest power is called the H.C.F. of the expressions.

(b) Least Common Multiple (L.C.M.):

If an algebraic expression p(x) is exactly divisible by two or more expressions, then p(x) is called the Common Multiple of the given expressions. The Least Common Multiple (L.C.M.) is the product of common factors together with non-common factors of the given expressions.

(a) Finding H.C.F.:

We can find H. C. F. of given expressions by the following two methods.

(i) By Factorization **(ii)** By Division

Sometimes it is difficult to find factors of given expressions. In that case,

method of division can be used to find H. C. F. We consider some examples to explain these two methods.

(i) H.C.F. by Factorization:

Example: Find the H. C. F. of the following polynomials.

$$x^2-4,\ x^2+4x+4,\ 2x^2+x-6.$$

Solution: $x^2-4=(x+2)(x-2)$

$x^2+4x+4=(x+2)^2$

$2x^2+x-6=2x^2+4x-3x-6=2x(x+2)-3(x+2)=(x+2)(2x-3)$

Hence, H. C.F. = x+2.

(ii) H.C.F. by Division

Example: Use division method to find the H. C. F. of the polynomials

$$p(x)=x^3-7x^2+14x-8 \text{ and } q(x)=x^3-7x+6$$

Solution:

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3-7x+6 & x^3-7x^2+14x-8 \\ & \underline{\pm x^3 \qquad \mp 7x \pm 6} \\ & -7x^2+21x-14 \end{array}$$

Here the remainder can be factorized as

$$-7x^2+21x-14=-7(x^2-3x+2)$$

We ignore −7 because it is not common to both the given polynomials and consider x^2-3x+2.

$$\begin{array}{r|l} & \quad x+3 \\ \hline x^2-3x+2 & x^3-0x^2-7x+6 \\ & \underline{-x^3 \mp 3x^2 \pm 2x} \\ & 3x^2-9x+6 \\ & \underline{-3x^2 \mp 9x \pm 6} \\ & \quad 0 \end{array}$$

Hence H. C. F. of p(x) and q(x) is x^2-3x+2

Observe that:

(i) In finding H. C. F. by division, if required, any expression can be multiplied by a suitable integer to avoid fraction.

(ii) In case we are given three polynomials, then as a first step we find H. C. F, of any two of them and then find the H. C. F. of this H. C. F. and the third polynomial.

(b) Finding L.C.M. by Factorization:

Working Rule to find L.C.M. of given Algebraic Expressions:

(i) Factorize the given expressions completely i.e. to simplest form.

(ii) Then the L.C.M. is obtained by taking the product of each factor appearing in

any of the given expressions, raised to the highest power with which that factor appears.

Example: Find the L.C.M. of $p(x) = 12(x^3 - y^3)$ and $q(x) = 8(x^3 - xy^2)$

Solution: By prime factorization of the given expressions, we have

$$p(x) = 12(x^3 - y^3) = 2^2 \times 3 \times (x - y)(x^2 + xy + y^2)$$

And $\quad q(x) = 8(x^3 - xy^2) = 8x(x^2 - y^2) = 2^3x(x + y)(x - y)$

Hence L.C.M. of p(x) and q(x) is

$$2^3 \times 3 \times x(x + y)(x - y)(x^2 + xy + y^2) = 24x(x + y)(x^3 - y^3)$$

Relation between H.C.F. and L.C.M.:

Example: By factorization, find (i) H.C.F. (ii) L.C.M. of $p(x) = 12(x^5 - x^4)$ and $q(x) = 8(x^4 - 3x^3 + 2x^2)$. Establish a relation between p(x), q(x) and H.C.F. and L.C.M. of the expressions p(x) and q(x).

Solution: Firstly, let us factorize completely the given expressions p(x) and q(x) into irreducible factors. We have

$$p(x) = 12(x^5 - x^4) = 12x^4(x - 1) = 2^2 \times 3\, x^4 (x - 1)$$

And $q(x) = 8(x^4 - 3x^3 + 2x^2) = 8x^2(x^2 - 3x + 2) = 2^3 x^2 (x - 1)(x - 2)$

H.C.F. of p(x) and $q(x) = 2^2 x^2 (x - 1) = 4x^2 (x - 1)$

L.C.M. of p(x) and $q(x) = 2^3 \times 3 \times x^4 (x - 1)(x - 2)$

Observe that:

$$p(x)\, g(x) = 12x^4 (x - 1) \times 8x^2 (x - 1)(x - 2)$$
$$= 96x^6(x - 1)^2(x - 2) \qquad \text{...... (i)}$$

And (L.C.M.) (H.C.F.) $= [2^3 \times 3 \times x^4 (x - 1)(x - 2)]\,[4x^2 (x - 1)]$

$$= [24x^4 (x - 1)(x - 2)]\,[4x^2 (x - 1)]$$
$$= 96x^6(x-1)^2(x-2) \qquad \text{...... (ii)}$$

From (i) and (ii), it is clear that

$$L.C.M. \times H.C.F. = p(x) \times q(x)$$

Hence, if p(x), q(x) and one of H.C.F, or L.C.M. are known, we can find the unknown by the formulae,

I. **L.C.M.** $= \dfrac{p(x) \times q(x)}{H.C.F.}$ **or H.C.F.** $= \dfrac{p(x) \times q(x)}{L.C.M.}$

II. **If L.C.M., H.C.F. and one of p(x) or q(x) are known, then**

$$P(x) = \frac{L.C.M. \times H.C.F.}{q(x)} \text{ and } \quad q(x) = \frac{L.C.M. \times H.C.F.}{p(x)}$$

Note: L.C.M. and H.C.F. are unique except for a factor of (–I).

Example-1: Find H.C.F. of the polynomials, $p(x) = 20(2x^3 + 3x^2 - 2x)$, $q(x) = 9(5x^4 + 40x)$ Then using the above formula (I) find the L.C.M. of p(x) and q(x).

Solution: We have $p(x) = 20(2x^3 + 3x^2 - 2x) = 20x(2x^2 + 3x - 2)$

$$= 20x(2x^2 + 4x - x - 2) = 20x[2x(x + 2) - (x + 2)]$$
$$= 20x(x + 2)(2x - 1) = 2^2 \times 5 \times x(x + 2)(2x - 1)$$

$q(x) = 9(5x^4 + 40x) = 45x(x^3 + 8)$

$= 45x\,(x + 2)\,(x^2 - 2x + 4) = 5 \times 3^2 \times x\,(x + 2)\,(x^2 - 2x + 4)$

Thus H.C.F. of p(x) and q(x) = $5x(x + 2)$

Now L.C.M. $= \dfrac{p(x) \times q(x)}{\text{H.C.F.}}$

$$= \frac{2^2 \times 5 \times x(x+2)(2x-1) \times 5 \times 3^2 \times x(x+2)(x^2-2x+4)}{5x(x+2)}$$

$= 4 \times 5 \times 9 \times x(x + 2)\,(2x - 1)\,(x^2 - 2x + 4)$

$= 180x\,(x + 2)\,(2x - 1)\,(x^2 - 2x + 4)$

Example-2: Find the L.C.M. of

$p(x) = 6x^3 - 7x^2 - 27x + 8$ and $q(x) = 6x^3 + 17x^2 + 9x - 4$.

Solution: We have, by long division,

$$\begin{array}{r|l} & \quad 1 \\ 6x^3-7x^2-27x+8 & 6x^3+17x^2+9x-4 \\ & \underline{\pm 6x^3 \mp 7x^2 \mp 27x \pm 8} \\ & \quad 24x^2+36x-12 \end{array}$$

But the remainder

$24x^2 + 36x - 12 = 12(2x^2 + 3x - 1)$

Thus, ignoring 12, we have

$$\begin{array}{r|l} & \quad 3x-8 \\ 2x^2+3x-1 & 6x^3-7x^2-27x+8 \\ & \underline{\pm 6x^3 \pm 9x^2 \mp 3x} \\ & -16x^2-24x+8 \\ & \underline{\mp 16x^2 \pm 24x \pm 8} \\ & \qquad 0 \end{array}$$

Hence H.C.F. of p(x) and q(x) = $2x^2 + 3x - 1$

L.C.M. $= \dfrac{(6x^3-7x^2-27x+8)(6x^3+17x^2+9x-4)}{2x^2+3x-1}$

$= \dfrac{(6x^3-7x^2-27x+8)}{2x^2+3x-1} \times (6x^3+17x^2+9x-4)$

$= (3x - 8)\,(6x^3 + 17x^2 + 9x - 4)$

Application of H.C.F. and L.C.M.:

Example: The sum of two numbers is 120 and their H.C.F. is 12. Find the numbers.

Solution: Let the numbers be 12x and 12y, where x, y are numbers prime to each other.

Then $12x + 12y = 120 \quad \Rightarrow \quad x + y = 10$

Thus we have to find two numbers whose sum is 10. The possible such pairs of numbers are (1, 9), (2, 8), (3, 7), (4, 6), (5, 5).

The pairs of numbers which are prime to each other are (1, 9) and (3, 7), thus the required numbers are

$$1 \times 12, 9 \times 12, 3 \times 12, 7 \times 12$$

i.e., 12, 108 and 36, 84.

Solved Exercise 6.1

1. Find the H.C.F. of the following expressions.

(i) $39x^7y^3z$ and $91x^5y^6z^7$

Solution:
$$\begin{aligned} 39x^7y^3z &= 13\times3\times x^7y^3z \\ 91x^5y^6z^7 &= 13\times7\times x^5y^6z^7 \\ \text{H.C.F} &= 13x^5y^3z \end{aligned}$$

(ii) $102xy^2z$, $85x^2yz$ and $187xyz^2$

Solution:
$$\begin{aligned} 102xy^2z &= 17\times6\times xy^2z \\ 85x^2yz &= 17\times5\times x^2yz \\ 187xyz^2 &= 17\times11\times xyz^2 \\ \text{H.C.F} &= 17xyz \end{aligned}$$

2. Find the H.C.F. of the following expressions by factorization.

(i) x^2+5x+6, $x^2-4x-12$

Solution:
$$\begin{aligned} x^2+5x+6 &= x^2+3x+2x+6 \\ &= x(x+3)+2(x+3) \\ &= (x+2)(x+3) \\ x^2-4x-12 &= x^2-6x+2x-12 \\ &= x(x-6)+2(x-6) \\ &= (x-6)(x+2) \\ \text{H.C.F} &= (x+2) \end{aligned}$$

(ii) x^3-27, $x^2+6x-27$, $2x^2-18$

Solution:
$$\begin{aligned} x^3-27 &= (x)^3-(3)^3 \\ &= (x-3)\left[(x)^2+(x)(3)+(3)^2\right] \\ &= (x-3)(x^2+3x+9) \\ x^2+6x-27 &= x^2+9x-3x-27 \\ &= x(x+9)-3(x+9) \\ &= (x+9)(x-3) \\ 2x^2-18 &= 2(x^2-9) \\ &= 2\left[(x)^2-(3)^2\right] \end{aligned}$$

$= 2(x-3)(x+3)$

H.C.F $= (x-3)$

(iii) $x^3 - 2x^2 + x, x^2 + 2x - 3, x^2 + 3x - 4$

Solution: $x^3 - 2x^2 + x = x\left[x^2 - 2x + 1\right]$

$= x\left[(x)^2 - 2(x)(1) + (1)^2\right]$

$= x(x-1)^2$

$= x(x-1)(x-1)$

$x^2 + 2x - 3 = x^2 + 3x - x - 3$

$= x(x+3) - 1(x+3)$

$= (x+3)(x-1)$

$x^2 + 3x - 4 = x^2 + 4x - x - 4$

$= x(x+4) - 1(x+4)$

$= (x+4)(x-1)$

H.C.F $= (x-1)$

(iv) $18(x^3 - 9x^2 + 8x), 24(x^2 - 3x + 2)$

Solution: $18(x^3 - 9x^2 + 8x) = 18x\left[x^2 - 9x + 8\right]$

$= 18x\left[x^2 - 8x - x + 8\right]$

$= 18x\left[x(x-8) - 1(x-8)\right]$

$= 18x(x-8)(x-1)$

$24(x^2 - 3x + 2) = 24\left[x^2 - 2x - x + 2\right]$

$= 24\left[x(x-2) - 1(x-2)\right]$

$= 24(x-2)(x-1)$

H.C.F $= 6(x-1)$

(v) $36(3x^4 + 5x^3 - 2x^2), 54(27x^4 - x)$

Solution: $36(3x^4 + 5x^3 - 2x^2) = 36x^2\left[3x^2 + 5x - 2\right]$

$= 36x^2\left[3x^2 + 6x - x - 2\right]$

$= 36x^2\left[3x(x+2) - 1(x+2)\right]$

$= 36x^2(x+2)(3x-1)$

$54(27x^4 - x) = 54x\left[27x^3 - 1\right]$

$= 54x\left[(3x)^3 - (1)^3\right]$

$= 54x(3x-1)\left[(3x)^2 + (3x)(1) + (1)^2\right]$

$= 54x(3x-1)(9x^2 + 3x + 1)$

$$\text{H.C.F.} = 18x(3x-1)$$

3. Find the H.C.F of the following by division method.

(i) $x^3+3x^2-16x+12,\ x^3+x^2-10x+8$

Solution:

$$\begin{array}{r|l}
 & 1 \\
x^3+x^2-10x+8 & x^3+3x^2-16x+12 \\
 & \underline{\pm x^3 \pm x^2 \mp 10x \pm 8} \\
 & 2x^2-6x+4 \\
 & 2\left(x^2-3x+2\right)
\end{array}$$

Thus, ignoring 2, we have

$$\begin{array}{r|l}
 & x+4 \\
x^2-3x+2 & x^3+x^2-10x+8 \\
 & \underline{\pm x^3 \mp 3x^2 \pm 2x} \\
 & 4x^2-12x+8 \\
 & \underline{\pm 4x^2 \mp 12x \pm 8} \\
 & 0
\end{array}$$

Hence H.C.F. $= x^2-3x+2$

(ii) $x^4+x^3-2x^2+x-3\ ,\ 5x^3+3x^2-17x+6$

Solution:

$$\begin{array}{r|l}
 & x+2 \\
5x^3+3x^2-17x+6 & x^4+x^3-2x^2+x-3 \\
 & \underline{\times 5} \\
 & 5x^4+5x^3-10x^2+5x-15 \\
 & \underline{\pm 5x^4 \pm 3x^3 \mp 17x^2 \pm 6x} \\
 & 2x^3+7x^2-x-15 \\
 & \underline{\times 5} \\
 & 10x^3+35x^2-5x-75 \\
 & \underline{\pm 10x^3 \pm 6x^2 \mp 34x \pm 12} \\
 & 29x^2+29x-87 \\
 & 29\left(x^2+x-3\right)
\end{array}$$

Thus ignoring 29, we have

$$\begin{array}{r|l}
 & 5x-2 \\
x^2+x-3 & 5x^3+3x^2-17x+6 \\
 & \underline{\pm 5x^3 \pm 5x^2 \mp 15x} \\
 & -2x^2-2x+6 \\
 & \underline{\mp 2x^2 \mp 2x \pm 6} \\
 & 0
\end{array}$$

Hence H.C.F $= x^2+x-3$

(iii) $2x^5 - 4x^4 - 6x$, $x^5 + x^4 - 3x^3 - 3x^2$

Solution:

$$\begin{array}{r|l} & 2 \\ \hline x^5 + x^4 - 3x^3 - 3x^2 & 2x^5 - 4x^4 - 6x \\ & \pm 2x^5 \pm 2x^4 \mp 6x^3 \mp 6x^2 \\ \hline & -6x^4 + 6x^3 + 6x^2 - 6x \\ & -6(x^4 - x^3 - x^2 + x) \end{array}$$

Thus ignoring -6, we have

$$\begin{array}{r|l} & x + 2 \\ \hline x^4 - x^3 - x^2 + x & x^5 + x^4 - 3x^3 - 3x^2 \\ & \pm x^5 \mp x^4 \mp x^3 \pm x^2 \\ \hline & 2x^4 - 2x^3 - 4x^2 \\ & \pm 2x^4 \mp 2x^3 \mp 2x^2 \pm 2x \\ \hline & -2x^2 - 2x \\ & -2(x^2 + x) \end{array}$$

Thus ignoring -2, we have

$$\begin{array}{r|l} & x^2 - 2x + 1 \\ \hline x^2 + x & x^4 - x^3 - x^2 + x \\ & \pm x^4 \pm x^3 \\ \hline & -2x^3 - x^2 + x \\ & \mp 2x^3 \mp 2x^2 \\ \hline & x^2 + x \\ & \pm x^2 \pm x \\ \hline & 0 \end{array}$$

Hence H.C.F. $= x^2 + x = x(x + 1)$

4. Find the L.C.M. of the following expressions.

(i) $39x^7y^3z$ *and* $91x^5y^6z^7$

Solution: $39x^7y^3z = 3 \times 13x^7y^3z$

$91x^5y^6z^7 = 7 \times 13x^5y^6z^7$

L.C.M. $= 3 \times 7 \times 13x^7y^6z^7$

$= 273x^7y^6z^7$

(ii) $102xy^2z$, $85x^2yz$ *and* $187xyz^2$

Solution: $102xy^2z = 2 \times 3 \times 17xy^2z$

$85x^2yz = 5 \times 17x^2yz$

$$187xyz^2 = 11\times 17xyz^2$$

$$\text{L.C.M} = 2\times 3\times 5\times 11\times 17x^2y^2z^2 = 5610x^2y^2z^2$$

5. Find the L.C.M. of the following expressions by factorization.

(i) $x^2 - 25x + 100$ *and* $x^2 - x - 20$

Solution: $x^2 - 25x + 100 = x^2 - 20x - 5x + 100$

$$= x(x-20) - 5(x-20)$$
$$= (x-20)(x-5)$$

$$x^2 - x - 20 = x^2 - 5x + 4x - 20$$
$$= x(x-5) + 4(x-5)$$
$$= (x-5)(x+4)$$

$$\text{L.C.M.} = (x-5)(x-20)(x+4)$$

(ii) $x^2 + 4x + 4, x^2 - 4, 2x^2 + x - 6$

Solution: $x^2 + 4x + 4 = x^2 + 2x + 2x + 4$

$$= x(x+2) + 2(x+2)$$
$$= (x+2)(x+2)$$

$$x^2 - 4 = (x)^2 - (4)^2$$
$$= (x-2)(x+2)$$

$$2x^2 + x - 6 = 2x^2 + 4x - 3x - 6$$
$$= 2x(x+2) - 3(x+2)$$
$$= (x+2)(2x-3)$$

$$\text{L.C.M.} = (x+2)(x+2)(x-2)(2x-3)$$

(iii) $2(x^4 - y^4), 3(x^3 + 2x^2y - xy^2 - 2y^3)$

Solution: $2(x^4 - y^4) = 2\left[(x^2)^2 - (y^2)^2\right]$

$$= 2(x^2-y^2)(x^2+y^2) = 2(x-y)(x+y)(x^2+y^2)$$

$$3(x^3 + 2x^2y - xy^2 - 2y^3) = 3\left[x^2(x+2y) - y^2(x+2y)\right]$$
$$= 3(x+2y)(x^2-y^2) = 3(x-y)(x+y)(x+2y)$$

$$\text{L.C.M.} = 2\times 3(x-y)(x+y)(x^2+y^2)(x+2y)$$
$$= 6(x^2-y^2)(x^2+y^2)(x+2y) = 6(x^4-y^4)(x+2y)$$

(iv) $4(x^4 - 1), 6(x^3 - x^2 - x + 1)$

Solution: $4(x^4 - 1) = 2\times 2\left[(x^2)^2 - (1)^2\right]$

$$= 2\times 2\left[(x^2-1)(x^2+1)\right] = 2\times 2(x-1)(x+1)(x^2+1)$$

$$6(x^3-x^2-x+1) = 2\times3\left[x^2(x-1)-1(x-1)\right]$$
$$= 2\times3\left[(x-1)(x^2-1)\right] = 2\times3(x-1)(x+1)(x-1)$$

$$\text{L.C.M.} = 2\times2\times3(x-1)(x+1)(x^2+1)(x-1)$$
$$= 12(x^2-1)(x^2+1)(x-1) = 12(x^4-1)(x-1)$$

6. For what value of k is (x + 4) the H.C.F. of $x^2 + x - (2k+2)$ and $2x^2 + kx - 12$?

Solution:

$$\begin{array}{r|l} & \quad x-3 \\ \hline x+4 & x^2+x-(2k+2) \\ & \pm x^2 \pm 4x \\ \hline & -3x-(2k+2) \\ & \mp 3x \mp 12 \\ \hline & -(2k+2)+12 \end{array}$$

$$\Rightarrow -2k-2+12=0$$
$$-2k+10=0$$
$$-2k=-10$$
$$\Rightarrow k=5$$

Now as k = 5. So, $2x^2+kx-12=2x^2+5x-12$

$$\begin{array}{r|l} & \quad 2x-3 \\ \hline x+4 & 2x^2+5x-12 \\ & \pm 2x^2 \pm 8x \\ \hline & -3x-12 \\ & \mp 3x \mp 12 \\ \hline & 0 \end{array}$$

Hence $k=5$

7. If $(x+3)(x-2)$ is the H.C.F of $p(x)=(x+3)(2x^2-3x+k)$ and $q(x)=(x-2)(3x^2+7x-\ell)$, find k and ℓ.

Solution: To find k, we have

$$\frac{(x+3)(2x^2-3x+k)}{(x+3)(x-2)} = \frac{2x^2-3x+k}{x-2}$$

$$\begin{array}{r|l} & \quad 2x+1 \\ \hline x-2 & 2x^2-3x+k \\ & \pm 2x^2 \mp 4x \\ \hline & x+k \\ & \pm x \mp 2 \\ \hline & k+2 \end{array}$$

$$\Rightarrow k+2=0$$
$$k=-2$$

To find ℓ, we have

$$\frac{(x-2)(3x^2+7x-\ell)}{(x-2)(x+3)}=\frac{3x^2+7x-\ell}{x+3}$$

$$\begin{array}{r|l} & 3x-2 \\ \hline x+3 & 3x^2+7x-\ell \\ & \pm 3x^2 \pm 9x \\ \hline & -2x-\ell \\ & \mp 2x \mp 6 \\ \hline & -\ell+6 \end{array}$$

$$\Rightarrow \quad -\ell+6=0$$
$$-\ell=-6$$
$$\Rightarrow \quad \ell=6$$

8. The L.C.M. and H.C.F. of two polynomials p(x) and q(x) are $2(x^4-1)$ and $(x+1)(x^2+1)$ respectively. If $p(x)=x^3+x^2+x+1$, find q(x)

Solution:

L.C.M. $=2(x^4-1)$, H.C.F. $=(x+1)(x^2+1)$, $p(x)=x^3+x^2+x+1$, $q(x)=?$

We know that

$$p(x)\times q(x) = \text{L.C.M.}\times\text{H.C.F.}$$

$$q(x) = \frac{\text{L.C.M.}\times\text{H.C.F}}{p(x)}$$

$$=\frac{2(x^4-1)\times(x+1)(x^2+1)}{x^3+x^2+x+1}$$

$$=\frac{2(x-1)(x+1)(x^2+1)(x+1)(x^2+1)}{x^2(x+1)+1(x+1)}$$

$$=\frac{2(x-1)(x+1)(x^2+1)(x+1)(x^2+1)}{(x+1)(x^2+1)}$$

$$=2(x-1)(x+1)(x^2+1)$$

$$=2(x^2-1)(x^2+1)=2(x^4-1)$$

9. Let $p(x)=10(x^2-9)(x^2-3x+2)$ and $q(x)=10x(x+3)(x-1)^2$. If the H.C.F. of p(x), q(x) is 10(x+3) (x-1), find their L.C.M.

Solution: $p(x)=10(x^2-9)(x^2-3x+2), q(x)=10x(x+3)(x-1)^2$,

H.C.F = 10(x+3) (x-1), L.C.M. = ?

We know that

$$\text{L.C.M.} \times \text{H.C.F.} = p(x)\times q(x)$$

$$\text{L.C.M.} = \frac{p(x)\times q(x)}{\text{H.C.F.}}$$

$$= \frac{10(x^2-9)(x^2-3x+2)\times 10x(x+3)(x-1)^2}{10(x+3)(x-1)}$$

$$= \frac{10(x-3)(x+3)(x-1)(x-2)\times 10x(x+3)(x-1)^2}{10(x+3)(x-1)}$$

$$= 10(x-3)(x-2)\times x(x+3)(x-1)^2$$

$$= 10x(x-1)^2(x-2)(x-3)(x+3)$$

$$= 10x(x-1)^2(x-2)(x^2-9)$$

10. Let the product of L.C.M. and H.C.F. of two polynomials be $(x+3)^2(x-2)(x+5)$. If one polynomial is $(x+3)(x-2)$ and the second polynomial is $x^2+kx+15$, find the value of k.

Solution: Let $p(x)=(x+3)(x-2)$, $q(x) = x^2+kx+15$

$$\text{L.C.M.} \times \text{H.C.F.} = (x+3)^2(x-2)(x+5)$$

We know that

$$p(x)\times q(x) = \text{L.C.M.}\times\text{H.C.F.}$$

$$(x+3)(x-2)(x^2+kx+15) = (x+3)^2(x-2)(x+5)$$

$$\Rightarrow \quad x^2+kx+15 = (x+3)(x+5)$$

$$x^2+kx+15 = x^2+5x+3x+15$$

$$x^2+kx+15 = x^2+8x+15$$

$$\Rightarrow \quad k = 8$$

11. Waqas wishes to distribute 128 bananas and also 176 apples equally among a certain number of children. Find the highest number of children who can get the fruit in this way.

Solution: $128 = 2\times2\times2\times2\times2\times2\times2$

$176 = 2\times2\times2\times2\times11$

H.C.F $= 2\times2\times2\times2 = 16$ children

BASIC OPERATIONS ON ALGEBRAIC FRACTIONS

We shall now carryout the operations of sum, difference, product and division on algebraic fractions by giving some examples. We assume that all fractions are defined.

Example-1: Simplify $\frac{x+3}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}+\frac{x+1}{x^2-5x+6}$

Solution: $\dfrac{x+3}{x^2-3x+2}+\dfrac{x+2}{x^2-4x+3}+\dfrac{x+1}{x^2-5x+6}$

$$=\frac{x+3}{x^2-2x-x+2}+\frac{x+2}{x^2-3x-x+3}+\frac{x+1}{x^2-3x-2x+6}$$

$$=\frac{x+3}{x(x-2)-1(x-2)}+\frac{x+2}{x(x-3)-1(x-3)}+\frac{x+1}{x(x-3)-2(x-3)}$$

$$=\frac{x+3}{(x-2)(x-1)}+\frac{x+2}{(x-3)(x-1)}+\frac{x+1}{(x-3)(x-2)}$$

$$=\frac{(x+3)(x-3)+(x+2)(x-2)+(x+1)(x-1)}{(x-1)(x-2)(x-3)}$$

$$=\frac{x^2-9+x^2-4+x^2-1}{(x-1)(x-2)(x-3)}=\frac{3x^2-14}{(x-1)(x-2)(x-3)}$$

Example-2: Express the product $\dfrac{x^3-8}{x^2-4}\times\dfrac{x^2+6x+8}{x^2-2x+1}$ as an algebraic expression reduced to lowest forms.

Solution: By factorizing completely, we have

$$\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}=\frac{(x-2)(x^2+2x+4)\times(x+2)(x+4}{(x-2)(x+2)\times(x-1)^2} \quad \text{........ (i)}$$

Now the factors of numerator are $(x-2)$, (x^2+2x+4), $(x+2)$ and $(x+4)$ and the factors of denominator are $(x-2)$, $(x+2)$ and $(x-1)^2$.

Therefore, their H.C.F. is $(x-2)\,x(x+2)$.

By canceling H.C.F. i.e., $(x-2)(x+2)$ from (i), we get the simplified form of given product as the fraction $\dfrac{(x^2+2x+4)(x+4}{(x-1)^2}$.

Example-3: Divide $\dfrac{x^2+x+1}{x^2-9}$ by $\dfrac{x^3-1}{x^2-4x+3}$ and simplify by reducing to lowest forms.

Solution: We have $\dfrac{x^2+x+1}{x^2-9}\div\dfrac{x^3-1}{x^2-4x+3}$

$$=\frac{(x^2+x+1)}{(x^2-9)}\times\frac{(x^2-4x+3)}{(x^3-1)}$$

$$=\frac{(x^2+x+1)(x^2-x-3x+3)}{(x^2-9)(x^3-1)}$$

$$=\frac{(x^2+x+1)(x-3)(x-1)}{(x+3)(x-3)(x-1)(x^2+x+1)}=\frac{1}{x+3}$$

oooooooooooo

Solved Exercise 6.2

1. Simplify each of the following as a rational expression.

(i) $\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12}$

Solution: $\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12}=\frac{x^2-3x+2x-6}{(x)^2-(3)^2}+\frac{x^2+6x-4x-24}{x^2-4x+3x-12}$

$$=\frac{x(x-3)+2(x-3)}{(x-3)(x+3)}+\frac{x(x+6)-4(x+6)}{x(x-4)+3(x-4)}$$

$$=\frac{(x-3)(x+2)}{(x-3)(x+3)}+\frac{(x+6)(x-4)}{(x-4)(x+3)}$$

$$=\frac{x+2}{x+3}+\frac{x+6}{x+3}$$

$$=\frac{(x+2)+(x+6)}{x+3}$$

$$=\frac{x+2+x+6}{x+3}=\frac{2x+8}{x+3}=\frac{2(x+4)}{x+3}$$

2. $\left[\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$

Solution:

$$\left[\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}=\left[\frac{(x+1)^2-(x-1)^2}{(x-1)(x+1)}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{(x^2+2x+1)-(x^2-2x+1)}{x^2-1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{x^2+2x+1-x^2+2x-1}{x^2-1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{4x}{x^2-1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\frac{4x(x^2+1)-4x(x^2-1)}{(x^2-1)(x^2+1)}+\frac{4x}{x^4-1}$$

$$=\frac{4x^3+4x-4x^3+4x}{x^4-1}+\frac{4x}{x^4-1}$$

$$=\frac{8x}{x^4-1}+\frac{4x}{x^4-1}=\frac{8x+4x}{x^4-1}=\frac{12x}{x^4-1}$$

3. $\frac{1}{x^2-8x+15}+\frac{1}{x^2-4x+3}-\frac{2}{x^2-6x+5}$

Solution: $\frac{1}{x^2-8x+15}+\frac{1}{x^2-4x+3}-\frac{2}{x^2-6x+5}$

$$=\frac{1}{x^2-3x-5x+15}+\frac{1}{x^2-1x-3x+3}-\frac{2}{x^2-5x-x+5}$$

$$=\frac{1}{(x-3)(x-5)}+\frac{1}{(x-1)(x-3)}-\frac{2}{(x-1)(x-5)}$$

$$=\frac{(x-1)+(x-5)-2(x-3)}{(x-1)(x-3)(x-5)}=\frac{x-1+x-5-2x+6}{(x-1)(x-3)(x-5)}$$

$$=\frac{2x-2x+6-6}{(x-1)(x-3)(x-5)}=\frac{0}{(x-1)(x-3)(x-5)}=0$$

4. $\frac{(x+2)(x+3)}{x^2-9}+\frac{(x+2)(2x^2-32)}{(x-4)(x^2-x-6)}$

Solution: $\frac{(x+2)(x+3)}{(x-3)(x+3)}+\frac{(x+2)2(x^2-16)}{(x-4)(x-3)(x+2)}$

$$=\frac{(x+2)(x+3)(x+2)(x-4)+2(x+2)(x-4)(x+4)(x+3)}{(x+2)(x-3)(x+3)(x-4)}$$

$$=\frac{(x+2)(x+3)(x-4)[(x+2)+2(x+4)]}{(x+2)(x-3)(x+3)(x-4)}=\frac{x+2+2x+8}{(x-3)}=\frac{3x+10}{x-3}$$

5. $\frac{x+3}{2x^2+9x+9}+\frac{1}{2(2x-3)}-\frac{4x}{4x^2-9}$

Solution: $\frac{x+3}{2x^2+9x+9}+\frac{1}{2(2x-3)}-\frac{4x}{4x^2-9}$

$$=\frac{x+3}{2x^2+3x+6x+9}+\frac{1}{2(2x-3)}-\frac{4x}{(2x)^2-(3)^2}$$

$$=\frac{x+3}{(x+3)(2x+3)}+\frac{1}{2(2x-3)}-\frac{4x}{(2x-3)(2x+3)}$$

$$=\frac{2(2x-3)(x+3)+(2x+3)(x+3)-2x4\times(x+3)}{2(2x+3)(2x-3)(x+3)}$$

$$=\frac{\cancel{(x+3)}[2(2x-3)+(2x+3)-2\times 4x]}{2(2x+3)(2x-3)\cancel{(x+3)}}$$

$$=\frac{2(2x-3)+(2x+3)-8x}{2(2x+3)(2x-3)} \quad =\frac{4x-6+2x+3-8x}{2(2x+3)(2x-3)}$$

$$=\frac{-2x-3}{2(2x+3)(2x-3)}$$

$$=\frac{-\cancel{(2x+3)}}{2\cancel{(2x+3)}(2x-3)}$$

$$=\frac{-1}{2(2x-3)}$$

6. $A-\frac{1}{A}$ *where* $A=\frac{a+1}{a-1}$

Solution: As $A=\frac{a+1}{a-1} \Rightarrow \frac{1}{A}=\frac{a-1}{a+1}$

$$A-\frac{1}{A}=\frac{a+1}{a-1}-\frac{a-1}{a+1}$$

$$=\frac{(a+1)^2-(a-1)^2}{(a-1)(a+1)}$$

$$=\frac{(a^2+2a+1)-(a^2-2a+1)}{(a^2-1)}$$

$$=\frac{a^2+2a+1-a^2+2a-1}{(a^2-1)}=\frac{4a}{a^2-1}$$

7. $\left[\frac{x-1}{x-2}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{4-x^2}\right]$

Solution: $\left[\frac{x-1}{x-2}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{4-x^2}\right]$

$$=\left[\frac{x-1}{x-2}-\frac{2}{x-2}\right]-\left[\frac{x+1}{x+2}-\frac{4}{x^2-4}\right]$$

$$=\left[\frac{x-1-2}{x-2}\right]-\left[\frac{(x+1)(x-2)-4}{(x-2)(x+2)}\right]$$

$$=\left[\frac{x-3}{x-2}\right]-\left[\frac{x^2+x-2x-2-4}{(x-2)(x+2)}\right]$$

$$=\frac{x-3}{x-2}-\frac{x^2-x-2-4}{(x-2)(x+2)}$$

$$=\frac{x-3}{x-2}-\frac{x^2-x-6}{(x-2)(x+2)}$$

$$=\frac{x-3}{x-2}-\frac{(x+2)(x-3)}{(x-2)(x+2)}=\frac{x-3}{x-2}-\frac{x-3}{x-2}=0$$

8. What rational expression should be subtracted from $\frac{2x^2+2x-7}{x^2+x-6}$ to get $\frac{x-1}{x-2}$?

Solution: $\frac{2x^2+2x-7}{x^2+x-6}-\frac{x-1}{x-2}$

$$=\frac{2x^2+2x-7}{(x-2)(x+3)}-\frac{x-1}{x-2}$$

$$=\frac{(2x^2+2x-7)-(x-1)(x+3)}{(x-2)(x+3)}=\frac{(2x^2+2x-7)-(x^2+3x-x-3)}{(x-2)(x+3)}$$

$$=\frac{2x^2+2x-7-x^2-3x+x+3}{(x-2)(x+3)}=\frac{2x^2+2x-7-x^2-2x+3}{(x-2)(x+3)}$$

$$=\frac{x^2-4}{(x-2)(x+3)}=\frac{(x-2)(x+2)}{(x-2)(x+3)}=\frac{x+2}{x+3}$$

Perform the indicated operations and simplify to the lowest form.

9. $\frac{x^2+x-6}{x^2-x-6}\times\frac{x^2-4}{x^2-9}$

Solution: $$\frac{x^2+x-6}{x^2-x-6}\times\frac{x^2-4}{x^2-9}=\frac{(x-2)(x+3)}{(x+2)(x-3)}\times\frac{(x-2)(x+2)}{(x-3)(x+3)}$$

$$=\frac{(x-2)(x-2)}{(x-3)(x-3)}=\frac{(x-2)^2}{(x-3)^2}$$

10. $\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}$

Solution: $\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}$

$$=\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}\times\frac{(x+2)(x+4)}{(x-1)(x-1)}$$

$$=\frac{(x+4)(x^2+2x+4)}{(x-1)^2}$$

11. $\frac{x^4-8x}{2x^2+5x-3}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x^2-2x}$

Solution: $\frac{x^4-8x}{2x^2+5x-3}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x^2-2x}$

$$=\frac{x(x^3-8)}{(2x-1)(x+3)}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x(x-2)}$$

$$=\frac{x(x-2)(x^2+2x+4)}{(2x-1)(x+3)}\times\frac{(2x-1)}{(x^2+2x+4)}\times\frac{x+3}{x(x-2)}=1$$

12. $\frac{2y^2+7y-4}{3y^2-13y+4}\div\frac{4y^2-1}{6y^2+y-1}$

Solution: $\frac{2y^2+7y-4}{3y^2-13y+4}\div\frac{4y^2-1}{6y^2+y-1}$

$$=\frac{2y^2+7y-4}{3y^2-13y+4}\times\frac{6y^2+y-1}{4y^2-1}$$

$$=\frac{(2y-1)(y+4)}{(3y-1)(y-4)}\times\frac{(3y-1)(2y+1)}{(2y-1)(2y+1)}=\frac{y+4}{y-4}$$

13. $\left[\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right]\div\left[\frac{x+y}{x-y}-\frac{x-y}{x+y}\right]$

Solution: $\left[\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right]\div\left[\frac{x+y}{x-y}-\frac{x-y}{x+y}\right]$

$$=\left[\frac{(x^2+y^2)^2-(x^2-y^2)^2}{(x^2-y^2)(x^2+y^2)}\right]\div\left[\frac{(x+y)^2-(x-y)^2}{(x-y)(x+y)}\right]$$

$$=\left[\frac{(x^4+2x^2y^2+y^4)-(x^4-2x^2y^2+y^4)}{(x^2-y^2)(x^2+y^2)}\right]\div\left[\frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{(x^2-y^2)}\right]$$

$$=\left[\frac{x^4+2x^2y^2+y^4-x^4+2x^2y^2-y^4}{(x^2-y^2)(x^2+y^2)}\right]\div\left[\frac{x^2+2xy+y^2-x^2+2xy-y^2}{(x^2-y^2)}\right]$$

$$=\left[\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}\right]\div\left[\frac{4xy}{(x^2-y^2)}\right]=\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}\times\frac{(x^2-y^2)}{4xy}$$

$$=\frac{xy}{x^2+y^2}$$

SQUARE ROOT OF ALGEBRAIC EXPRESSION

Definition:

As with numbers we define the square root of a given expression p(x) as another expression q(x) such that q(x) . q(x) = p(x).

As 5 × 5 =25, so square root of 25 is 5.

It means we can find square root of the expression p(x) if it can be expressed as a perfect square.

In this section we shall find square root of an algebraic expression

(i) by factorization **(ii)** by division.

(i) By factorization:

Example-1: Use factorization to find the square root of the expression:

$$4x^2 - 12x + 9$$

Solution: $4x^2 - 12x + 9 = 4x^2 - 6x - 6x + 9 = 2x(2x-3) - 3(2x-3)$

$= (2x-3)(2x-3) = (2x-3)^2$

Hence, $\sqrt{4x^2 - 12x + 9} = \pm(2x-3)$

Example-2: Find the square root of $x^2 + \frac{1}{x^2} + 12\left(x + \frac{1}{x}\right) + 38, x \neq 0$

Solution: $x^2 + \frac{1}{x^2} + 12\left(x + \frac{1}{x}\right) + 38$

$= x^2 + \frac{1}{x^2} + 2 + 12\left(x + \frac{1}{x}\right) + 36$ (adding and subtracting 2)

$= \left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right)(6) + (6)^2$

$= \sqrt{x^2 + \frac{1}{x^2} + 12\left(x + \frac{1}{x}\right)^{+38}} = \sqrt{\left[\pm\left(x + \frac{1}{x} + 6\right)\right]^2} = \pm\left(x + \frac{1}{x} + 6\right)$

Hence the required square root is $\pm\left(x + \frac{1}{x} + 6\right)$

(ii) By Division:

When it is difficult to convert the given expression into a perfect square by factorization, we use the method of actual division to find its square root. The method is similar to the division method of finding square root of numbers.

Example-1: Find the square root of $4x^4 + 12x^3 + x^2 - 12x + 4$

Solution: We note that the given expression is already in descending order. Now the square root of the first term i.e., $\sqrt{4x^4} = 2x^2$. So the first term of the divisor and quotient will be $2x^2$ in the first step. At each successive step, the remaining terms will be brought down.

$$\begin{array}{r|l} & 2x^2+3x-2 \\ \hline 2x^2 & 4x^4+12x^3+x^2-12x+4 \\ & \pm 4x^4 \\ \hline 4x^2+3x & 12x^3+x^2-12x+4 \\ & \pm 12x^3 \pm 9x^2 \\ \hline 4x^2+6x-2 & -8x^2-12x+4 \\ & \mp 8x^2 \mp 12x \pm 4 \\ \hline & 0 \end{array}$$

Thus square root of given expression is $\pm(2x^2 + 3x - 2)$

Example-2: Find the square root of the expression

$$4\frac{x^2}{y^2} + 8\frac{x}{y} + 16 + 12\frac{y}{x} + 9\frac{y^2}{x^2}$$

Solution: We note that the given expression is in descending powers of x.

Now $\sqrt{4\frac{x^2}{y^2}} = 2\frac{x}{y}$. So proceeding as usual, we have

$$\begin{array}{r|l} & 2\frac{x}{y}+2+3\frac{y}{x} \\ \hline 2\frac{x}{y} & 4\frac{x^2}{y^2}+8\frac{x}{y}+16+12\frac{y}{x}+9\frac{y^2}{x^2} \\ & \pm 4\frac{x^2}{y^2} \\ \hline 4\frac{x}{y}+2 & 8\frac{x}{y}+16 \\ & \pm 8\frac{x}{y} \pm 4 \\ \hline 4\frac{x}{y}+4+3\frac{y}{x} & 12+12\frac{y}{x}+9\frac{y^2}{x^2} \\ & \pm 12 \pm 12\frac{y}{x} \pm 9\frac{y^2}{x^2} \\ \hline & 0 \end{array}$$

Hence the square root of given expression is $\pm\left(2\frac{x}{y}+2+3\frac{y}{x}\right)$

Example-3: To make the expression $x^4 - 10x^3 + 33x^2 - 42x + 20$ a perfect square,

(i) What should be added to it?

(ii) What should be subtracted from it?

(iii) What should be the value of x?

Solution:

$$\begin{array}{r|l}
 & x^2-5x+4 \\
x^2 & x^4-10x^3+33x^2-42x+20 \\
 & \pm x^4 \\
2x^2-5x & -10x^3+33x^2 \\
 & \mp 10x^3 \pm 25x^2 \\
2x^2-10+4 & 8x^2-42x+20 \\
 & \pm 8x^2 \mp 40x \pm 16 \\
 & -2x+4
\end{array}$$

For making the given expression a perfect square the remainder must be zero. Hence

(i) we should add $(2x-4)$ to the given expression

(ii) we should subtract $(-2x+4)$ from the given expression

(iii) we should take $-2x+4=0$ to find the value of x. This gives the required value of x i.e., $x=2$.

Solved Exercise 6.3

1. Use factorization to find the square root of the following expressions.

(i) $4x^2-12xy+9y^2$

Solution:
$$\begin{aligned}4x^2-12xy+9y^2 &= 4x^2-6xy-6xy+9y^2 \\ &= 2x(2x-3y)-3y(2x-3y) \\ &= (2x-3y)(2x-3y) \\ &= (2x-3y)^2\end{aligned}$$

Taking square root on both sides, we get
$$\sqrt{4x^2-12xy+9y^2}=\pm\sqrt{(2x-3y)^2}=\pm(2x-3y)$$

(ii) $x^2-1+\frac{1}{4x^2}(x\neq 0)$

Solution: $x^2-1+\frac{1}{4x^2}=(x)^2-2(x)\left(\frac{1}{2x}\right)+\left(\frac{1}{2x}\right)^2=\left(x-\frac{1}{2x}\right)^2$

Taking square root on both sides, we get
$$\sqrt{x^2-1+\frac{1}{4x^2}}=\pm\sqrt{\left(x-\frac{1}{2x}\right)^2}=\pm\left(x-\frac{1}{2x}\right)$$

(iii) $\frac{1}{16}x^2-\frac{1}{12}xy+\frac{1}{36}y^2$

Solution: $\frac{1}{16}x^2-\frac{1}{12}xy+\frac{1}{36}y^2=\left(\frac{1}{4}x\right)^2-2\left(\frac{1}{4}x\right)\left(\frac{1}{6}y\right)+\left(\frac{1}{6}y\right)^2$

$$=\left(\frac{1}{4}x-\frac{1}{6}y\right)^2=\left(\frac{x}{4}-\frac{y}{6}\right)^2$$

Taking square root on both sides, we get

$$\sqrt{\frac{1}{16}x^2-\frac{1}{12}xy+\frac{1}{36}y^2}=\pm\sqrt{\left(\frac{x}{4}-\frac{y}{6}\right)^2}=\pm\left(\frac{x}{4}-\frac{y}{6}\right)$$

(iv) $4(a+b)^2-12(a^2-b^2)+9(a-b)^2$

Solution:

$$4(a+b)^2-12(a^2-b^2)+9(a-b)^2=[2(a+b)]^2-2[2(a+b)]\,[3(a-b)]+[3(a-b)]^2$$
$$=[2(a+b)-3(a-b)]^2$$
$$=[2a+2b-3a+3b]^2=(5b-a)^2$$

$$\sqrt{4(a+b)^2-12(a^2-b^2)+9(a-b)^2}=\pm\sqrt{(5b-a)^2}=\pm(5b-a)$$

(v) $\dfrac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}$

Solution: $$\frac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}=\frac{(2x^3)^2-2(2x^3)(3y^3)+(3y^3)^2}{(3x^2)^2+2(3x^2)(4y^2)+(4y^2)^2}=\frac{(2x^3-3y^3)^2}{(3x^2+4y^2)^2}$$

Taking square root on both sides, we get

$$\sqrt{\frac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}}=\pm\sqrt{\frac{(2x^3-3y^3)^2}{(3x^2-4y^2)^2}}=\pm\frac{(2x^3-3y^3)}{(3x^2+4y^2)}$$

(vi) $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right), (x\neq 0)$

Solution: $$\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)=x^2+\frac{1}{x^2}+2-4\left(x-\frac{1}{x}\right)$$
$$=x^2+\frac{1}{x^2}-4\left(x-\frac{1}{x}\right)+2$$

By adding & subtracting "2", we get

$$=\left(x^2+\frac{1}{x^2}-2\right)-4\left(x-\frac{1}{x}\right)+2+2$$
$$=\left(x-\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)+4$$
$$=\left(x-\frac{1}{x}\right)^2-2\left(x-\frac{1}{x}\right)(2)+(2)^2=\left[\left(x-\frac{1}{x}\right)-2\right]^2$$

Taking square root on both sides, we get

$$\sqrt{\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)}=\pm\sqrt{\left[\left(x-\frac{1}{x}\right)-2\right]^2}=\pm\left[\left(x-\frac{1}{x}\right)-2\right]$$

(vii) $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12 (x\neq 0)$

Solution: $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}+2\right)+12$

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)-8+12$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-2\left(x^2+\frac{1}{x^2}\right)(2)+(2)^2$$

$$=\left[\left(x^2+\frac{1}{x^2}\right)-2\right]^2$$

Taking square root on both sides, we get

$$\sqrt{\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)+12}=\pm\sqrt{\left[\left(x^2+\frac{1}{x^2}\right)-2\right]^2}=\pm\left[\left(x^2+\frac{1}{x^2}\right)-2\right]$$

(viii) $(x^2+3x+2)(x^2+4x+3)(x^2+5x+6)$

Solution: $(x^2+3x+2)(x^2+4x+3)(x^2+5x+6)$

$$=\left[x^2+2x+x+2\right]\left[x^2+3x+x+3\right]\left[x^2+3x+2x+6\right]$$

$$=\left[x(x+2)+1(x+2)\right]\left[x(x+3)+1(x+3)\right]\left[x(x+3)+2(x+3)\right]$$

$$=(x+2)(x+1)(x+3)(x+1)(x+3)(x+2)$$

$$=(x+1)^2(x+2)^2(x+3)^2$$

Taking square root on both sides, we get

$$\sqrt{(x^2+3x+2)(x^2+4x+3)(x^2+5x+6)}=\pm\sqrt{(x+1)^2(x+2)^2(x+3)^2}$$

$$=\pm(x+1)(x+2)(x+3)$$

(ix) $(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21)$

Solution: $(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21)$

$$=\left[x^2+7x+x+7\right]\left[2x^2-3x+2x-3\right]\left[2x^2+14x-3x-21\right]$$

$$=\left[x(x+7)+1(x+7)\right]\left[x(2x-3)+1(2x-3)\right]\left[2x(x+7)-3(x+7)\right]$$

$$=(x+7)(x+1)(2x-3)(x+1)(x+7)(2x-3)$$
$$=(x+1)^2(x+7)^2(2x-3)^2$$

Taking square root on both sides, we get

$$\sqrt{(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21)}=\pm\sqrt{(x+1)^2(x+7)^2(2x-3)^2}$$
$$=\pm(x+1)(x+7)(2x-3)$$

2. Use division method to find the square root of the following expressions.

(i) $4x^2+12xy+9y^2+16x+24y+16$

Solution:

$$\begin{array}{r|l} & 2x+3y+4 \\ \hline 2x & 4x^2+12xy+9y^2+16x+24y+16 \\ & \pm 4x^2 \\ \hline 4x+3y & 12xy+9y^2+16x+24y+16 \\ & \pm 12xy \pm 9y^2 \\ \hline 4x+6y+4 & 16x+24y+16 \\ & \pm 16x \pm 24y \pm 16 \\ \hline & 0 \end{array}$$

Hence the square root of given expression is
$\pm(2x+3y+4)$

(ii) $x^4-10x^3+37x^2-60x+36$

Solution:

$$\begin{array}{r|l} & x^2-5x+6 \\ \hline x^2 & x^4-10x^3+37x^2-60x+36 \\ & \pm x^4 \\ \hline 2x^2-5x & -10x^3+37x^2-60x+36 \\ & \mp 10x^3 \pm 25x^2 \\ \hline 2x^2-10x+6 & 12x^2-60x+36 \\ & \pm 12x^2 \mp 60x \pm 36 \\ \hline & 0 \end{array}$$

Hence the square root of given expression is $\pm(x^2-5x+6)$

(iii) $9x^4-6x^3+7x^2-2x+1$

Solution:

$$\begin{array}{r|l} & 3x^2-x+1 \\ 3x^2 & 9x^4-6x^3+7x^2-2x+1 \\ & \pm 9x^4 \\ 6x^2-x & -6x^3+7x^2-2x+1 \\ & \mp 6x^3 \pm x^2 \\ 6x^2-2x+1 & 6x^2-2x+1 \\ & \pm 6x^2 \mp 2x \pm 1 \\ & 0 \end{array}$$

Hence the square root of given expression is $\pm\left(3x^2-x+1\right)$

(iv) $4+25x^2-12x-24x^3+16x^4$

Solution:

$$\begin{array}{r|l} & 4x^2-3x+2 \\ 4x^2 & 16x^4-24x^3+25x^2-12x+4 \\ & \pm 16x^4 \\ 8x^2-3x & -24x^3+25x^2-12x+4 \\ & \mp 24x^3 \pm 9x^2 \\ 8x^2-6x+2 & 16x^2-12x+4 \\ & \pm 16x^2 \mp 12x \pm 4 \\ & 0 \end{array}$$

Hence the square root of given expression is $\pm\left(4x^2-3x+2\right)$

(v) $\frac{x^2}{y^2}-10\frac{x}{y}+27-10\frac{y}{x}+\frac{y^2}{x^2}$ $(x\neq 0, y\neq 0)$

Solution:

$$\begin{array}{r|l} & \frac{x}{y}-5+\frac{y}{x} \\ \frac{x}{y} & \frac{x^2}{y^2}-10\frac{x}{y}+27-10\frac{y}{x}+\frac{y^2}{x^2} \\ & \pm\frac{x^2}{y^2} \\ 2\frac{x}{y}-5 & -10\frac{x}{y}+27-10\frac{y}{x}+\frac{y^2}{x^2} \\ & \mp 10\frac{x}{y} \pm 25 \\ 2\frac{x}{y}-10+\frac{y}{x} & 2-10\frac{y}{x}+\frac{y^2}{x^2} \\ & \pm 2 \mp 10\frac{y}{x} \pm \frac{y^2}{x^2} \\ & 0 \end{array}$$

Hence the square root of given expression is

$$\pm\left(\frac{x}{y}-5-\frac{y}{x}\right)$$

3. Find the value of k for which the following expressions will become a perfect square.

(i) $4x^4-12x^3+37x^2-42x+k$

Solution:

$$\begin{array}{r|l} & 2x^2-3x+7 \\ \hline 2x^2 & 4x^4-12x^3+37x^2-42x+k \\ & \pm 4x^4 \\ \hline 4x^2-3x & -12x^3+37x^2-42x+k \\ & \mp 12x^3 \pm 9x^2 \\ \hline 4x^2-6x+7 & 28x^2-42x+k \\ & \pm 28x^2 \mp 42x \pm 49 \\ \hline & k-49 \end{array}$$

$$\Rightarrow k-49=0$$

$$k=49$$

(ii) $x^4-4x^3+10x^2-kx+9$

Solution:

$$\begin{array}{r|l} & x^2-2x+3 \\ \hline x^2 & x^4-4x^3+10x^2-kx+9 \\ & \pm x^4 \\ \hline 2x^2-2x & -4x^3+10x^2-kx+9 \\ & \mp 4x^3 \pm 4x^2 \\ \hline 2x^2-4x+3 & 6x^2-kx+9 \\ & \pm 6x^2 \mp 12x \pm 9 \\ \hline & -kx+12x \end{array}$$

$$\Rightarrow \quad -kx+12x=0$$

$$-kx=-12x$$

$$\Rightarrow \quad k=12$$

4. Find the values of ℓ and m for which the following expressions will become perfect squares.

(i) $x^4 + 4x^3 + 16x^2 + lx + m$

Solution:

$$\begin{array}{l} \qquad\qquad x^2 + 2x + 6 \\ x^2 \overline{)\, x^4 + 4x^3 + 16x^2 + lx + m} \\ \quad \pm x^4 \\ 2x^2 + 2x \overline{)\, 4x^3 + 16x^2 + lx + m} \\ \qquad\quad \pm 4x^3 \pm 4x^2 \\ 2x^2 + 4x + 6 \overline{)\, \qquad 12x^2 + lx + m} \\ \qquad\qquad\qquad \pm 12x^2 \pm 24x \pm 36 \\ \qquad\qquad\qquad (lx - 24x) + (m - 36) \end{array}$$

$$\Rightarrow \quad lx - 24x = 0 \text{ and } m - 36 = 0$$

$$lx = 24x \qquad m = 36$$

$$\Rightarrow \quad l = 24$$

(ii) $49x^4 - 70x^3 + 109x^2 + lx - m$

Solution:

$$\begin{array}{l} \qquad\qquad 7x^2 - 5x + 6 \\ 7x^2 \overline{)\, 49x^4 - 70x^3 + 109x^2 + \ell x - m} \\ \quad \pm 49x^4 \\ 14x^2 - 5x \overline{)\, -70x^3 + 109x^2 + \ell x - m} \\ \qquad\quad \mp 70x^3 \pm 25x^2 \\ 14x^2 - 10x + 6 \overline{)\, \qquad 84x^2 + \ell x - m} \\ \qquad\qquad\qquad \pm 84x^2 \mp 60x \pm 36 \\ \qquad\qquad\qquad (\ell x + 60x) - m - 36 \end{array}$$

$$\Rightarrow \quad \ell x + 60x = 0 \text{ and } -m - 36 = 0$$

$$\ell x = -60x \qquad -m = 36$$

$$\Rightarrow \quad \ell = -60 \qquad m = -36$$

5. To make the expression $9x^4 - 12x^3 + 22x^2 - 13x + 12$, a perfect square

(i) what should be added to it?

(ii) what should be subtracted from it?

(iii) what should be the value of x?

Solution:

$$
\begin{array}{r|l}
 & 3x^2-2x+3 \\
3x^2 & 9x^4-12x^3+22x^2-13x+12 \\
 & \pm 9x^4 \\
6x^2-2x & -12x^3+22x^2-13x+12 \\
 & \mp 12x^3 \pm 4x^2 \\
6x^2-4x+3 & 18x^2-13x+12 \\
 & \pm 18x^2 \mp 12x \pm 9 \\
 & -x+3
\end{array}
$$

(i) $-(-x+3)=x-3$ should be added to it.

(ii) $-x+3$ should be subtracted from it.

(iii) $-x+3=0 \quad \Rightarrow \quad -x=-3 \quad \Rightarrow \quad x=3$

SOLVED REVIEW EXERCISE 6

1. Choose the correct answer.

(i) H.C.F. of p^3q-pq^3 and $p^5q^2-p^2q^5$ is ......

(a) $pq(p^2-q^2)$ (b) $pq(p-q)$ (c) $p^2q^2(p-q)$ (d) $pq(p^3-q^3)$

(ii) H.C.F. of $5x^2y^2$ and $20x^3y^3$ is ......

(a) $5x^2y^2$ (b) $20x^3y^3$ (c) $100x^5y^5$ (d) $5xy$

(iii) H.C.F. of $x-2$ and x^2+x-6 is ......

(a) x^2+x-6 (b) $x+3$ (c) $x-2$ (d) $x+2$

(iv) H.C.F. of a^3+b^3 and a^2-ab+b^2 is......

(a) $a+b$ (b) a^2-ab+b^2 (c) $(a-b)^2$ (d) a^2+b^2

(v) H.C.F. of x^2-5x+6 and x^2-x-6 is......

(a) $x-3$ (b) $x+2$ (c) x^2-4 (d) $x-2$

(vi) H.C.F. of a^2-b^2 and a^3-b^3 is ......

(a) $a-b$ (b) $a+b$ (c) a^2+ab+b^2 (d) a^2-ab+b^2

(vii) H.C.F. of x^2+3x+2, x^2+4x+3, and x^2+5x+4 is ....

(a) $x+1$ (b) $(x+1)(x+2)$ (c) $x+3$ (d) $(x+4)(x+1)$

(viii) L.C.M. of $15x^2$, $45xy$ and $30xyz$ is ....

(a) $90xyz$ (b) $90x^2yz$ (c) $15xyz$ (d) $15x^2yz$

(ix) L.C.M. of a^2+b^2 and a^4-b^4 is ....

(a) a^2+b^2 (b) a^2-b^2 (c) a^4-b^4 (d) $a-b$

(x) The product of two algebraic expressions is equal to the ____ of their H.C.F. and L.C.M.

(a) Sum (b) Difference (e) Product (d) Quotient

(xi) **Simplify** $\frac{a}{9a^2-b^2}+\frac{1}{3a-b}=$__________

(a) $\frac{4a}{9a^2-b^2}$ (b) $\frac{4a-b}{9a^2-b^2}$ (c) $\frac{4a+b}{9a^2-b^2}$ (d) $\frac{b}{9a^2-b^2}$

(xii) **Simplify** $\frac{a^2+5a-14}{a^2-3a-18}\times\frac{a+3}{a-2}=$__________

(a) $\frac{a+7}{a-6}$ (b) $\frac{a+7}{a-2}$ (c) $\frac{a+3}{a-6}$ (d) $\frac{a-2}{a+3}$

(xiii) **Simplify** $\frac{a^3-b^3}{a^4-b^4}\div\frac{a^2+ab+b^2}{a^2+b^2}=$..............

(a) $\frac{1}{a+b}$ (b) $\frac{1}{a-b}$ (c) $\frac{a-b}{a^2+b^2}$ (d) $\frac{a+b}{a^2+b^2}$

(xiv) **Simplify** $\left(\frac{2x+y}{x+y}-1\right)\div\left(1-\frac{x}{x+y}\right)=$__________

(a) $\frac{x}{x+y}$ (b) $\frac{y}{x+y}$ (c) $\frac{y}{x}$ (d) $\frac{x}{y}$

(xv) **The square root of a^2-2a+1 is ....**

(a) $\pm(a+1)$ (b) $\pm(a-1)$ (c) $a-1$ (d) $a+1$

(xvi) **What should be added to complete the square of x^4+64 ?.........**

(a) $8x^2$ (b) $-8x^2$ (c) $16x^2$ (d) $4x^2$

(xvii) **The square root of $x^4+\frac{1}{x^4}+2$ is ....**

(a) $\pm\left(x+\frac{1}{x}\right)$ (b) $\pm\left(x^2+\frac{1}{x^2}\right)$ (c) $\pm\left(x-\frac{1}{x}\right)$ (d) $\pm\left(x^2-\frac{1}{x^2}\right)$

Solution:

(i) b	(ii) a	(iii) c	(iv) b	(v) a	(vi) a	(vii) a
(viii) b	(ix) c	(x) c	(xi) c	(xii) a	(xiii) a	(xiv) d
(xv) b	(xvi) c	(xvii) b				

2. **Find the H. C . F of the following by factorization.** $8x^4-128,\quad 12x^3-96$

Solution:

$$8x^4-128 = 8(x^4-16) = 2\times2\times2\left[(x^2)^2-(4)^2\right] = 2\times2\times2(x^2-4)(x^2+4)$$
$$= 2\times2\times2(x+2)(x-2)(x^2+4)$$
$$12x^3-96 = 12(x^3-8) = 2\times2\times3\left[(x)^3-(2)^3\right] = 2\times2\times3(x-2)(x^2+2x+4)$$
$$\text{H.C.F.} = 2\times2\times(x-2) = 4(x-2)$$

3. **Find the H.C.F. of the following by division method.**

$y^3+3y^2-3y-9,\ y^3+3y^2-8y-24$

Solution:

$$\begin{array}{r|l} & 1 \\ y^3+3y^2-3y-9 & y^3+3y^2-8y-24 \\ & \underline{\pm y^3 \pm 3y^2 \mp 3y \mp 9} \\ & -5y-15 \end{array}$$

$$= -5(y+3)$$

Thus ignoring -5, we have

$$\begin{array}{r|l} & y^2 \\ y+3 & y^3+3y^2-3y-9 \\ & \underline{y^3+3y^2} \\ & -3y-9 \\ & -3(y+3) \end{array}$$

Ignoring -3 we have

$$\begin{array}{r|l} & 1 \\ y+3 & y+3 \\ & \underline{y+3} \\ & 0 \end{array}$$

H.C.F. = y+3

4. Find the L.C.M. of the following by factorization.

$12x^2-75, 6x^2-13x-5, 4x^2-20x+25$

Solution: $12x^2-75 = 3(4x^2-25) = 3\left[(2x)^2-(5)^2\right]$

$$= 3(2x-5)(2x+5)$$

$$6x^2-13x-5 = 6x^2-15x+2x-5 = 3x(2x-5)+1(2x-5) = (2x-5)(3x+1)$$

$$4x^2-20x+25 = 4x^2-10x-10x+25 = 2x(2x-5)-5(2x-5) = (2x-5)(2x-5)$$

$$\text{L.C.M.} = 3(2x+5)(3x+1)(2x-5)^2$$

5. If H.C.F of $x^4+3x^3+5x^2+26x+56$ and $x^4+2x^3-4x^2-x+28$ is x^2+5x+7, find their L.C.M.

Solution: $p(x) = x^4+3x^3+5x^2+26x+56, q(x) = x^4+2x^3-4x^2-x+28$

H.C.F $= x^2+5x+7$, L.C.M. = ?

We know that L.C.M. × H.C.F. $= p(x) \times q(x)$

$$\text{L.C.M.} = \frac{p(x) \times q(x)}{\text{H.C.F.}}$$

$$= \frac{(x^4+3x^3+5x^2+26x+56) \times (x^4+2x^3-4x^2-x+28)}{x^2+5x+7}$$

Now

$$
\begin{array}{r|l}
 & x^2-2x+8 \\
\hline
x^2+5x+7 & x^4+3x^3+5x^2+26x+56 \\
 & \underline{\pm x^4 \pm 5x^3 \pm 7x^2} \\
 & -2x^3-2x^2+26x+56 \\
 & \underline{\mp 2x^3 \mp 10x^2 \mp 14x} \\
 & 8x^2+40x+56 \\
 & \underline{\pm 8x^2 \pm 40x \pm 56} \\
 & 0
\end{array}
$$

So, $\text{L.C.M.} = \left(x^2-2x+8\right)\left(x^4+2x^3-4x^2-x+28\right)$

6. Simplify

(i) $\dfrac{3}{x^3+x^2+x+1}-\dfrac{3}{x^3-x^2+x-1}$

Solution: $\dfrac{3}{x^3+x^2+x+1}-\dfrac{3}{x^3-x^2+x-1}$

$$=\frac{3}{(x+1)(x^2+1)}-\frac{3}{(x-1)(x^2+1)}$$

$$=\frac{3(x-1)-3(x+1)}{(x-1)(x+1)(x^2+1)}=\frac{3x-3-3x-3}{(x^2-1)(x^2+1)}=\frac{6}{1-x^4}$$

(ii) $\dfrac{a+b}{a^2-b^2}+\dfrac{a^2-ab}{a^2-2ab+b^2}$

Solution: $\dfrac{a+b}{a^2-b^2}\div\dfrac{a^2-ab}{a^2-2ab+b^2}$

$$=\frac{a+b}{a^2-b^2}\times\frac{a^2-2ab+b^2}{a^2-ab}=\frac{a+b}{(a+b)(a-b)}\times\frac{(a-b)^2}{a(a-b)}=\frac{1}{a}$$

7. Find square root by using factorization.

$\left(x^2+\dfrac{1}{x^2}\right)+10\left(x+\dfrac{1}{x}\right)+27\ (x\neq 0)$

Solution: $\left(x^2+\dfrac{1}{x^2}\right)+10\left(x+\dfrac{1}{x}\right)+27=\left(x^2+\dfrac{1}{x^2}\right)+10\left(x+\dfrac{1}{x}\right)+25+2$

$$=\left(x^2+\frac{1}{x^2}+2\right)+10\left(x+\frac{1}{x}\right)+25$$

$$=\left(x+\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)(5)+(5)^2=\left[\left(x+\frac{1}{x}\right)+5\right]^2$$

Taking square root on both sides, we get

$$\sqrt{\left(x^2+\frac{1}{x^2}\right)+10\left(x+\frac{1}{x}\right)+27}=\pm\sqrt{\left[\left(x+\frac{1}{x}\right)+5\right]^2}=\pm\left[\left(x+\frac{1}{x}\right)+5\right]$$

8. Find square root by using division method.

$$\frac{4x^2}{y^2}+\frac{20x}{y}+13-\frac{30y}{x}+\frac{9y^2}{x^2}\ (x,y\neq 0)$$

Solution:

$$\begin{array}{l|l}
 & \frac{2x}{y}+5-\frac{3y}{x} \\
\hline
\frac{2x}{y} & \frac{4x^2}{y^2}+\frac{20x}{y}+13-\frac{30y}{x}+\frac{9y^2}{x^2} \\
 & \pm\frac{4x^2}{y^2} \\
\hline
\frac{4x}{y}+5 & \frac{20x}{y}+13-\frac{30y}{x}+\frac{9y^2}{x^2} \\
 & \pm\frac{20x}{y}\pm 25 \\
\hline
\frac{4x}{y}+10-\frac{3y}{x} & -12-\frac{30y}{x}+\frac{9y^2}{x^2} \\
 & \mp 12\mp\frac{30y}{x}\pm\frac{9y^2}{x^2} \\
\hline
 & 0
\end{array}$$

Hence the square root of given expression is $\pm\left(\frac{2x}{y}+5-\frac{3y}{x}\right)$

SUMMARY

- We learned to find the H.C.F. and L.C.M. of algebraic expressions by the methods of factorization and division.
- We established a relation between H.C.F. and L.C.M. of two polynomials p(x) and q(x) given by the formula

  $$\text{L.C.M.}\times\text{H.C.F.}=p(x)\times q(x)$$

  and used it to determine L.C.M. or H.C.F. etc.
- Any unknown expression may be found if three of them are known by using relation.

  $$\text{L.C.M}\times\text{H.C.F}=p(x)\times q(x)$$
- H.C.F. and L.C.M. are used to simplify fractional expressions involving basic operations of +, −, x, ÷.
- Determination of square root of algebraic expression by factorization and division methods has defined and explained.

UNIT 7

LINEAR EQUATIONS AND INEQUALITIES

Unit Outlines	
7.1 *Linear Equations*	7.2 *Equations involving Absolute Value*
7.3 *Linear Inequalities*	7.4 *Solving Linear Inequalities*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- recall linear equation in one variable.
- solve linear equation with rational coefficients.
- reduce equations, involving radicals, to simple linear form and find their solutions.
- define absolute value.
- solve the equation, involving absolute value, in one variable.
- define inequalities (>, <) and (≥, ≤).
- recognize properties of inequalities (i.e., trichotomy, transitive, additive and multiplicative).
- solve linear inequalities with rational coefficients.

Definition:

A linear equation in one variable x is an equation of the form.

$$ax + b = 0, \text{ where } a, b \in R \text{ and } a \neq 0.$$

A solution to a linear equation is any replacement or substitution for the variable x that makes the statement true. Two linear equations are said to be equivalent if they have exactly the same solution. The variable x occurring in the equation is also called **the unknown.**

Solving a Linear Equation in One Variable:

The process of solving an equation involves finding a sequence of equivalent equations until the variable x is isolated on one side of the equation to give the solution.

Technique for Solving:

The procedure for solving linear equations in one variable is summarized in the following box.

- If fractions are present, we multiply each side by the L.C.M. of the denominators to eliminate them.
- To remove parentheses we use to distributive property.
- Combine alike terms, if any, on both sides.
- Use the addition property of equality (add or subtract) to get all the variables

on one side and constants on the other side.

- Use the multiplication property of equality to isolate the variable on one side.
- Check the answer by replacing the variable in the original equation.

Example-1: Solve the equation $\frac{3x}{2}-\frac{x-2}{3}=\frac{25}{6}$

Solution: Multiplying each side of the given equation by the L.C.M. 6 to eliminate fractions, we get.

$$9x-2(x-2)=25$$
$$9x-2x+4=25$$
$$7x=21$$
$$x=3$$

Check: Substituting x = 3 in original equation, we get

$$\frac{3}{2}(3)-\frac{3-2}{3}=\frac{25}{6}$$
$$\frac{9}{2}-\frac{1}{3}=\frac{25}{6}$$
$$\frac{25}{6}=\frac{25}{6} \quad \text{(which is true)}$$

Since x = 3 makes the original statement true, therefore the solution is correct.

Example-2: Solve $\frac{3}{y-1}-2=\frac{3y}{y-1}$.

Solution: To clear fractions we multiply both sides by the L.C.M. = y – 1 and get

$$\frac{3}{y-1}-2=\frac{3y}{y-1}$$
$$3-2(y-1)=3y$$
$$3-2y+2=3y$$
$$-5y=-5$$
$$y=1$$

Check: Substituting y = 1 in the given equation, we have

$$\frac{3}{1-1}-2=\frac{3(1)}{1-1}$$
$$\frac{3}{0}-2=\frac{3}{0}$$

But $\frac{3}{0}$ is undefined, so y = 1 cannot be solution.

Thus the given equation has no solution.

Example-3: Solve $\frac{3x-1}{3}-\frac{2x}{x-1}=x$.

Solution: To clear fractions we multiply both sides by 3 (x – 1) with the assumption

that

$x - 1 \neq 0$ i.e., $x \neq 1$, and get

$$\frac{3x-1}{3} - \frac{2x}{x-1} = x$$

$$(x-1)(3x-1) - 6x = 3x(x-1)$$

$$3x^2 - 4x + 1 - 6x = 3x^2 - 3x$$

$$-10x + 1 = -3x$$

$$-7x = -1 \quad \Rightarrow \quad x = \frac{1}{7}$$

Check: On substituting $x = \frac{1}{7}$ the original equation is verified a true statement. That means the restriction $x \neq 1$ has no effect on the solution because $\frac{1}{7} \neq 1$.

Hence our solution $x = \frac{1}{7}$ is correct.

Equations Involving Radicals but Reducible to Linear Form:

Definition:

When the variable in an equation occurs under a radical, the equation is called a **radical equation.**

The procedure to solve a radical equation is to eliminate the radical by raising each side to a power equal to the index of the radical.

When raising each side of the equation to a certain power may produce a nonequivalent equation that has more solutions than the original equation. These additional solutions are called extraneous solutions. We must check our answer(s) for such solutions when working with radical equations.

Example-1: Solve the equations:

(a) $\sqrt{2x-3} - 7 = 0$ **(b)** $\sqrt[3]{3x+5} = \sqrt[3]{x-1}$

Solution (a): To isolate the radical, we can rewrite the given equation as

$$\sqrt{2x-3} = 7 \quad \Rightarrow \quad 2x - 3 = 49 \Rightarrow \quad 2x = 49 + 3$$

$$2x = 52 \quad \Rightarrow \quad x = 26$$

Check: Let us substitute $x = 26$ in the original equation. Then

$$\sqrt{2(26)-3} - 7 = 0 \quad \Rightarrow \quad \sqrt{52-3} - 7 = 0$$

$$\sqrt{49} - 7 = 0 \quad \Rightarrow \quad 0 = 0$$

Hence, solution set = $\{26\}$

Solution (b): $\sqrt[3]{3x+5} = \sqrt[3]{x-1} \quad \Rightarrow \quad 3x + 5 = x - 1$

$$2x = -6 \quad \Rightarrow \quad x = -3$$

Check: We substitute $x = -3$ in the original equation. Then

$$\sqrt[3]{3(-3)+5} = \sqrt[3]{-3-1} \quad \Rightarrow \quad \sqrt[3]{-4} = \sqrt[3]{-4}$$

Thus $x = -3$ satisfies the original equation.

Here $\sqrt[3]{-4}$ is a real number because we raised each side of the equation to an odd power.

Hence, solution set = $\{-3\}$

Example-2: Solve and check: $\sqrt{5x-7}-\sqrt{x+10}=0$

Solution: When two terms of a radical equation contain variables in the radicand, we express the equation such that one of these terms is on each side. So we rewrite the equation in this form to get.

$$\sqrt{5x-7}-\sqrt{x+10}=0 \quad\Rightarrow\quad \sqrt{5x-7}=\sqrt{x+10}$$
$$5x-7=x+10 \quad\Rightarrow\quad 4x=17 \quad\Rightarrow\quad x=\frac{17}{4}$$

Check: Substituting $x=\frac{17}{4}$ in original equation.

$$\sqrt{5x-7}-\sqrt{x+10}=0 \quad\Rightarrow\quad \sqrt{5\left(\frac{17}{4}\right)-7}-\sqrt{\frac{17}{4}+10}=0$$
$$\sqrt{\frac{57}{4}}-\sqrt{\frac{57}{4}}=0 \quad\Rightarrow\quad 0=0$$

i.e., $x=\frac{17}{4}$ makes the given equation a true statement.

Hence, solution set = $\left\{\frac{17}{4}\right\}$.

Example-3: Solve $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

Solution: $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

Squaring both sides, we get

$$x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$$
$$2\sqrt{x^2+9x+14}=4x+4$$
$$\sqrt{x^2+9x+14}=2x+2$$

Again squaring both sides, we get

$$x^2+9x+14=4x^2+8x+4$$
$$3x^2-x-10=0$$
$$3x^2-6x+5x-10=0$$
$$3x(x-2)+5(x-2)=0$$
$$(x-2)(3x+5)=0$$
$$\Rightarrow (x-2)=0 \quad\text{or}\quad (3x+5)=0$$
$$x=2 \qquad\qquad x=-\frac{5}{3}$$

On checking, we see that $x=2$ satisfy the equation, but $x=-\frac{5}{3}$ does not satisfy the equation. So solution set is $\{2\}$ and $x=-\frac{5}{3}$ is called an extraneous root.

Solved Exercise 7.1

1. Solve the following equations.

(i) $\frac{2}{3}x-\frac{1}{2}x=x+\frac{1}{6}$

Solution: $\frac{2}{3}x-\frac{1}{2}x=x+\frac{1}{6}$

$$6\times\frac{2}{3}x-6\times\frac{1}{2}x=6\times x+6\times\frac{1}{6}$$
$$4x-3x=6x+1$$
$$x=6x+1$$
$$x-6x=1$$
$$-5x=1 \quad \Rightarrow \quad x=-\frac{1}{5}$$

Check: Substituting $x=-\frac{1}{5}$ in original equation, we get

$$\frac{2}{3}\left(-\frac{1}{5}\right)-\frac{1}{2}\left(-\frac{1}{5}\right)=-\frac{1}{5}+\frac{1}{6}$$
$$-\frac{2}{15}+\frac{1}{10}=-\frac{1}{5}+\frac{1}{6}$$

On multiplying both sides by 30, we get

$$30\times\left(-\frac{2}{15}\right)+30\times\frac{1}{10}=30\times\left(-\frac{1}{5}\right)+30\times\frac{1}{6}$$
$$-4+3=-6+5$$
$$-1=-1 \quad \text{(which is true)}$$

Since $x=\frac{-1}{5}$ makes the original statement true, therefore the solution is correct.

Hence the solution set $=\{-\frac{1}{5}\}$

(ii) $\frac{x-3}{3}-\frac{x-2}{2}=-1$

Solution: $\frac{x-3}{3}-\frac{x-2}{2}=-1$

On multiplying both sides by 6, we get

$$6\times\frac{x-3}{3}-6\times\frac{x-2}{2}=6\times(-1)$$
$$2(x-3)-3(x-2)=-6$$
$$2x-6-3x+6=-6$$

$$-x = -6$$

$$\Rightarrow \quad x = 6$$

Check: Substituting $x = 6$ in original equation, we get

$$\frac{6-3}{3}-\frac{6-2}{2}=-1$$

$$\frac{3}{3}-\frac{4}{2}=-1$$

$$1-2=-1$$

$$-1=-1 \quad \text{(which is true)}$$

Since $x = 6$ makes the original statement true, therefore the solution is correct.
Hence the solution set $= \{6\}$

(iii) $\frac{1}{2}\left(x-\frac{1}{6}\right)+\frac{2}{3}=\frac{5}{6}+\frac{1}{3}\left(\frac{1}{2}-3x\right)$

Solution: $\frac{1}{2}\left(x-\frac{1}{6}\right)+\frac{2}{3}=\frac{5}{6}+\frac{1}{3}\left(\frac{1}{2}-3x\right)$

$$\frac{1}{2}x-\frac{1}{12}+\frac{2}{3}=\frac{5}{6}+\frac{1}{6}-x$$

$$12\times\frac{1}{2}x-12\times\frac{1}{12}+12\times\frac{2}{3}=12\times\frac{5}{6}+12\times\frac{1}{6}-12\times x$$

$$6x-1+8=10+2-12x$$

$$6x+12x=12-7$$

$$18x=5$$

$$x=\frac{5}{18}$$

Check: Substituting $x=\frac{5}{18}$ in original equation, we get

$$\frac{1}{2}\left(\frac{5}{18}-\frac{1}{6}\right)+\frac{2}{3}=\frac{5}{6}+\frac{1}{3}\left(\frac{1}{2}-3\times\frac{5}{18}\right)$$

$$\frac{5}{36}-\frac{1}{12}+\frac{2}{3}=\frac{5}{6}+\frac{1}{6}-\frac{5}{18}$$

$$36\times\frac{5}{36}-36\times\frac{1}{12}+36\times\frac{2}{3}=36\times\frac{5}{6}+36\times\frac{1}{6}-36\times\frac{5}{18}$$

$$5-3+24=30+6-10$$

$$-3+29=36-10$$

$$26=26 \quad \text{(which is true)}$$

Since $x=\frac{5}{18}$ makes the original statement true. Therefore the solution is correct.
Hence the solution set $=\left\{\frac{5}{18}\right\}$

(iv) $x+\frac{1}{3}=2\left(x-\frac{2}{3}\right)-6x$

Solution:

$$x+\frac{1}{3}=2\left(x-\frac{2}{3}\right)-6x$$
$$x+\frac{1}{3}=2x-\frac{4}{3}-6x$$
$$3x+3\times\frac{1}{3}=3\times 2x-3x\frac{4}{3}-3\times 6x$$
$$3x+1=6x-4-18x$$
$$3x+1=-12x-4$$
$$3x+12x=-1-4$$
$$3x+12x=-1-4$$
$$15x=-5$$
$$x=-\frac{1}{3}$$

Check: Substituting $x=-\frac{1}{3}$ in original equation, we get.

$$-\frac{1}{3}+\frac{1}{3}=2\left(-\frac{1}{3}-\frac{2}{3}\right)-6\left(-\frac{1}{3}\right)$$
$$0=2\left(\frac{-1-2}{3}\right)+2$$
$$0=2\left(\frac{-3}{3}\right)+2$$
$$0=-2+2$$
$$0=0 \qquad \text{(which is true)}$$

Since $x=-\frac{1}{3}$ makes the original statement true. Therefore the solution is correct.

Hence the solution set $=\left\{-\frac{1}{3}\right\}$

(v) $\frac{5(x-3)}{6}-x=1-\frac{x}{9}$

Solution: $\frac{5(x-3)}{6}-x=1-\frac{x}{9}$

$$18\times\frac{5(x-3)}{6}-18x=18-18\times\frac{x}{9}$$
$$3\times 5(x-3)-18x=18-2x$$
$$15(x-3)-18x=18-2x$$

$$15x-45-18x=18-2x$$
$$-3x-45=18-2x$$
$$-3x+2x=18+45$$
$$-x=63$$
Or $$x=-63$$

Check: Substituting x = – 63 in original equation, we get

$$\frac{5(-63-3)}{6}-(-63)=1-\frac{(-63)}{9}$$
$$\frac{5(-66)}{6}+63=1+7$$
$$5(-11)+63=8$$
$$-55+63=8$$
$$8=8 \text{ (which is true)}$$

Since x = – 63 makes the original statement true. Therefore the solution is correct.
Hence the solution set = $\{-63\}$

(vi) $\frac{x}{3x-6}=2-\frac{2x}{x-2},\ x\neq 2$

Solution:
$$\frac{x}{3x-6}=2-\frac{2x}{x-2}$$
$$\frac{x}{3x-6}=\frac{2(x-2)-2x}{x-2}$$
$$\frac{x}{3x-6}=\frac{2x-4-2x}{x-2}$$
$$\frac{x}{3x-6}=\frac{-4}{x-2}$$
$$x(x-2)=-4(3x-6)$$
$$x^2-2x=-12x+24$$
$$x^2-2x+12x-24=0$$
$$x^2+10x-24=0$$
$$x^2+12x-2x-24=0$$
$$x(x+12)-2(x+12)=0$$
$$(x+12)(x-2)=0$$
$$\Rightarrow x-2=0 \quad \text{or} \quad x+12=0$$
$$x=2 \qquad\qquad x=-12$$

Check: Substituting x = 2 in original equation, we get

$$\frac{2}{3(2)-6}=2-\frac{2(2)}{2-2}$$

$$\frac{2}{6-6}=2-\frac{4}{0}$$
$$\frac{2}{0}=2-\infty$$
$$\infty \neq 2 \qquad \text{(which is not true)}$$

Since x = 2 does not make the original statement true, therefore the solution is not correct.

Substituting x = – 12 in original equation, we get.

$$\frac{-12}{3(-12)-6}=2-\frac{2(-12)}{-12-2}$$
$$\frac{-12}{-36-6}=2-\frac{-24}{-14}$$
$$\frac{-12}{-42}=2-\frac{24}{14}$$
$$\frac{2}{7}=\frac{28-24}{14}$$
$$\frac{2}{7}=\frac{4}{14}$$
$$\frac{2}{7}=\frac{2}{7} \quad \text{(which is true)}$$

Since x = – 12 makes the original statement true, therefore the solution is correct.

Hence the solution set = $\{-12\}$

(vii) $\dfrac{2x}{2x+5}=\dfrac{2}{3}-\dfrac{5}{4x+10}, x \neq -\dfrac{5}{2}$

Solution: $\dfrac{2x}{2x+5}=\dfrac{2}{3}-\dfrac{5}{4x+10}$

$$\frac{2x}{2x+5}=\frac{2(4x+10)-5(3)}{3(4x+10)}$$
$$\frac{2x}{2x+5}=\frac{8x+20-15}{12x+30}$$
$$\frac{2x}{2x+5}=\frac{8x+5}{12x+30}$$
$$2x(12x+30)=(8x+5)(2x+5)$$
$$24x^2+60x=16x^2+40x+10x+25$$
$$24x^2-16x^2+60x-50x-25=0$$
$$8x^2+10x-25=0$$

$$8x^2+20x-10x-25=0$$
$$4x(2x+5)-5(2x+5)=0$$
$$(2x+5)(4x-5)=0$$
$$\Rightarrow \quad 2x+5=0 \quad \text{or} \quad 4x-5=0$$
$$2x=-5 \qquad 4x=5$$
$$x=-\frac{5}{2} \qquad x=\frac{5}{4}$$

Check: Substituting $x=\frac{5}{4}$ in original equation, we get

$$\frac{2\left(\frac{5}{4}\right)}{2\left(\frac{5}{4}\right)+5}=\frac{2}{3}-\frac{5}{4\left(\frac{5}{4}\right)+10} \quad \Rightarrow \quad \frac{\frac{5}{2}}{\frac{5}{2}+5}=\frac{2}{3}-\frac{5}{5+10}$$

$$\frac{\frac{5}{2}}{\frac{15}{2}}=\frac{2}{3}-\frac{5}{15} \quad \Rightarrow \quad \frac{5}{2}\times\frac{2}{15}=\frac{2}{3}-\frac{1}{3}$$

$$\frac{1}{3}=\frac{1}{3} \qquad \text{(which is true)}$$

Since $x=\frac{5}{4}$ makes the original statement true, therefore the solution is correct.

Hence the solution set $=\frac{5}{4}$

Substituting $x=\frac{-5}{2}$ in original equation, we get

$$\frac{2\left(\frac{-5}{2}\right)}{2\left(\frac{-5}{2}\right)+5}=\frac{2}{3}-\frac{5}{4\left(\frac{-5}{2}\right)+10} \quad \Rightarrow \quad \frac{-5}{-5+5}=\frac{2}{3}-\frac{5}{-10+10}$$

$$\frac{-5}{0}=\frac{2}{3}-\frac{5}{0}$$

$$\infty=\infty \qquad \text{(which is not true)}$$

Since $x=\frac{-5}{2}$ does not make the original statement true, therefore the solution is not correct.

Hence, the solution set $=\left\{\frac{5}{4}\right\}$

(*viii*) $\frac{2x}{x-1}+\frac{1}{3}=\frac{5}{6}+\frac{2}{x-1}, \; x\neq 1$

Solution: $\frac{2x}{x-1}+\frac{1}{3}=\frac{5}{6}+\frac{2}{x-1}$

$$\frac{3(2x)+(x-1)}{3(x-1)}=\frac{5(x-1)+2(6)}{6(x-1)}$$

$$\frac{6x+x-1}{3(x-1)}=\frac{5x-5+12}{6(x-1)}$$

$$\frac{7x-1}{3(x-1)}=\frac{5x+7}{6(x-1)}$$

$$(6x-6)(7x-1)=(3x-3)(5x+7)$$

$$42x^2-6x-42x+6=15x^2+21x-15x-21$$

$$42x^2-48x+6=15x^2+6x-21$$

$$42x^2-15x^2-48x-6x+6+21=0$$

$$27x^2-54x+27=0$$

$$27(x^2-2x+1)=0$$

$$\Rightarrow \quad x^2-2x+1=0$$

$$(x-1)^2=0$$

$$\Rightarrow \quad x-1=0$$

$$x=1$$

Check: Substituting x = 1 in original equation, we get

$$\frac{2(1)}{1-1}+\frac{1}{3}=\frac{5}{6}+\frac{2}{1-1}$$

$$\frac{2}{0}+\frac{1}{3}=\frac{5}{6}+\frac{2}{0}$$

$$\infty+\frac{1}{3}=\frac{5}{6}+\infty$$

$$\infty=\infty \quad \text{(which is not true)}$$

Since x = 1 does not make the original statement true, therefore the solution is not correct.

Hence, there is no solution.

(ix) $\frac{2}{x^2-1}-\frac{1}{x+1}=\frac{1}{x+1}, x\neq\pm1$

Solution: $\frac{2}{x^2-1}-\frac{1}{x+1}=\frac{1}{x+1}$

$$\frac{2}{(x-1)(x+1)}-\frac{1}{x+1}=\frac{1}{x+1}$$

$$\frac{2-(x-1)}{(x-1)(x+1)}=\frac{1}{x+1}$$

$$\frac{2-x+1}{x^2-1}=\frac{1}{x+1}$$

$$\frac{3-x}{x^2-1}=\frac{1}{x+1}$$

$$\Rightarrow \quad x^2-1=(3-x)(x+1)$$

$$x^2-1=3x+3-x^2-x$$

$$x^2+x^2-2x-3-1=0$$

$$2x^2-2x-4=0$$

$$2(x^2-x-2)=0$$

$$\Rightarrow \quad x^2-x-2=0$$

$$x^2-2x+x-2=0$$

$$x(x-2)+1(x-2)=0$$

$$(x-2)(x+1)=0$$

$$\Rightarrow \quad x-2=0 \text{ or } x+1=0$$

$$x=2 \qquad x=-1$$

Check:

Substituting $x = 2$ in original equation, we get

$$\frac{2}{(2)^2-1}-\frac{1}{2+1}=\frac{1}{2+1}$$

$$\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$$

$$\frac{1}{3}=\frac{1}{3} \qquad \text{(which is true)}$$

Since $x = 2$ makes the original statement true, therefore the solution is correct.

Substituting $x = -1$ in original equation, we get

$$\frac{2}{(-1)^2-1}-\frac{1}{-1+1}=\frac{1}{-1+1}$$

$$\frac{2}{0}-\frac{1}{0}=\frac{1}{0}$$

$$\infty=\infty \qquad \text{(which is not true)}$$

Since $x = -1$ does not make the original statement true,

Hence, solution set= $\{2\}$

(x) $\frac{2}{3x+6}=\frac{1}{6}-\frac{1}{2x+4},\ x\neq -2$

Solution:
$$\frac{2}{3x+6}=\frac{1}{6}-\frac{1}{2x+4}$$
$$\frac{2}{3x+6}=\frac{2x+4-6}{6(2x+4)}$$
$$\frac{2}{3x+6}=\frac{2x-2}{12x+24}$$
$$(3x+6)(2x-2)=2(12x+24)$$
$$6x^2-6x+12x-12=24x+48$$
$$6x^2+6x-12-24x-48=0$$
$$6x^2-18x-60=0$$
$$6(x^2-3x-10)=0$$
$$\Rightarrow \quad x^2-3x-10=0$$
$$x^2-5x+2x-10=0$$
$$x(x-5)+2(x-5)=0$$
$$(x-5)(x+2)=0$$
$$\Rightarrow \quad x-5=0 \quad \text{or} \quad x+2=0$$
$$x=-2 \qquad x=5$$

Check: Substituting x = – 2 in original equation, we get
$$\frac{2}{3(-2)+6}=\frac{1}{6}-\frac{1}{2(-2)+4}$$
$$\frac{2}{-6+6}=\frac{1}{6}-\frac{1}{-4+4}$$
$$\frac{2}{0}=\frac{1}{6}-\frac{1}{0}$$
$$\infty=\infty \text{ (which is not true)}$$

Since x = – 2 does not make the original statement true, therefore solution is not correct.

Substituting x = 5 in original equation, we get
$$\frac{2}{3(5)+6}=\frac{1}{6}-\frac{1}{2(5)+4}$$
$$\frac{2}{15+6}=\frac{1}{6}-\frac{1}{10+4}$$
$$\frac{2}{21}=\frac{1}{6}-\frac{1}{14}$$

$$\frac{2}{21}=\frac{7-3}{42}$$
$$\frac{2}{21}=\frac{4}{42}$$
$$\frac{2}{21}=\frac{2}{21} \text{ (which is true)}$$

Since x = 5 makes the original statement true, therefore solution is correct.
Hence the solution set = {5}

2. Solve each equation and check for extraneous solution, if any.

(i) $\sqrt{3x+4}=2$

Solution: $\sqrt{3x+4}=2 \quad \Rightarrow \quad \left(\sqrt{3x+4}\right)^2=(2)^2$

$3x+4=4 \quad \Rightarrow \quad 3x=4-4$

$3x=0 \quad \Rightarrow \quad x=0$

Check: Substituting x = 0 in original equation, we get

$\sqrt{3(0)+4}=2 \quad \Rightarrow \quad \sqrt{4}=2$

$2=2$ (which is true)

Since x = 0 makes the statement true, therefore the solution is correct.
Hence the solution set = {0}

(ii) $\sqrt[3]{2x-4}-2=0$

Solution: $\sqrt[3]{2x-4}-2=0 \quad \Rightarrow \quad (2x-4)^{\frac{1}{3}}=2$

Taking cube on both sides, we get

$\left[(2x-4)^{\frac{1}{3}}\right]^3=(2)^3 \quad \Rightarrow \quad 2x-4=8$

$2x=8+4 \quad \Rightarrow \quad 2x=12 \quad \Rightarrow \quad x=6$

Check: Substituting x = 6 in the original equation, we get

$\sqrt[3]{2(6)-4}-2=0 \quad \Rightarrow \quad 3\sqrt{12-4}-2=0$

$\sqrt[3]{8}-2=0 \quad \Rightarrow \quad (2^3)^{\frac{1}{3}}-2=0$

$2-2=0 \quad \Rightarrow \quad 0=0$ (which is true)

Since x = 6 makes the statement true, therefore the solution is correct.
Hence the solution set = {6}

(iii) $\sqrt{x-3}-7=0$

Solution: $\sqrt{x-3}-7=0 \quad \Rightarrow \quad \sqrt{x-3}=7$

$\left(\sqrt{x-3}\right)^2=(7)^2$ (Squaring each side) $\Rightarrow \quad x-3=49$

$x=49+3 \quad \Rightarrow \quad x=52$

Check: Substituting $x = 52$ in the original equation, we get

$\sqrt{52-3}-7=0 \quad \Rightarrow \quad \sqrt{49}-7=0$

$7-7=0 \quad \Rightarrow \quad 0=0$ (which is true)

Since $x = 52$ makes the statement true, therefore the solution is correct.
Hence the solution set = $\{52\}$

(iv) $2\sqrt{t+4}=5$

Solution: $2\sqrt{t+4}=5 \quad \Rightarrow \quad \left(2\sqrt{t+4}\right)^2=(5)^2$ (Squaring each side)

$4(t+4)=25 \quad \Rightarrow \quad 4t+16=25$

$4t=25-16 \quad \Rightarrow \quad 4t=9 \quad \Rightarrow \quad t=\dfrac{9}{4}$

Check: Substituting $t=\dfrac{9}{4}$ in the original equation, we get

$2\sqrt{\dfrac{9}{4}+4}=5 \quad \Rightarrow \quad 2\sqrt{\dfrac{25}{4}}=5$

$2\left(\dfrac{5}{2}\right)=5 \quad \Rightarrow \quad 5=5$ (which is true)

Since $t=\dfrac{9}{4}$ makes the statement true, therefore the solution is correct.

Hence the solution set = $\{\dfrac{9}{4}\}$

(v) $\sqrt[3]{2x+3}=\sqrt[3]{x-2}$

Solution: $\sqrt[3]{2x+3}=\sqrt[3]{x-2} \quad \Rightarrow \quad (2x+3)^{\frac{1}{3}}=(x-2)^{\frac{1}{3}}$

$\left[(2x+3)^{\frac{1}{3}}\right]^3=\left[(x-2)^{\frac{1}{3}}\right]^3$ (taking cube of each side) $\Rightarrow \quad 2x+3=x-2$

$2x-x=-2-3 \quad \Rightarrow x=-5$

Check: Substituting $x=-5$ in the original equation, we get

$\sqrt[3]{2(-5)+3}=\sqrt[3]{-5-2} \quad \Rightarrow \quad (-10+3)^{\frac{1}{3}}=(-7)^{\frac{1}{3}}$

$\left[(-7)^{\frac{1}{3}}\right]^3=\left[(-7)^{\frac{1}{3}}\right]^3$ (taking cube of each side) $\Rightarrow \; -7=-7$ (which is true)

Since $x=-5$ makes the statement true, therefore the solution is correct.
Hence the solution set = $\{-5\}$

(v) $\sqrt[3]{2-t}=\sqrt[3]{2t-28}$

Solution: $\sqrt[3]{2-t}=\sqrt[3]{2t-28} \quad \Rightarrow \quad (2-t)^{\frac{1}{3}}=(2t-28)^{\frac{1}{3}}$

$\left[(2-t)^{\frac{1}{3}}\right]^3=\left[(2t-28)^{\frac{1}{3}}\right]^3$ (taking cube of each side) $\Rightarrow \quad 2-t=2t-28$

$-t-2t=-2-28 \quad \Rightarrow \quad -3t=-30$

$t=10$ which is true.

Check: Substituting $t=10$ in the original equation, we get

$\sqrt[3]{2-10}=\sqrt[3]{2(10)-28} \quad \Rightarrow \quad (-8)^{\frac{1}{3}}=(20-28)^{\frac{1}{3}}$

$\left[(-8)^{\frac{1}{3}}\right]^3=\left[(-8)^{\frac{1}{3}}\right]^3 \quad \Rightarrow \quad -8=-8$ which is true.

Since $t=10$ makes the statement true, therefore the solution is correct.

Hence the solution set $=\{10\}$

(vii) $\sqrt{2t+6}-\sqrt{2t-5}=0$

Solution: $\sqrt{2t+6}-\sqrt{2t-5}=0 \quad \Rightarrow \quad \sqrt{2t+6}=\sqrt{2t-5}$

$\left(\sqrt{2t+6}\right)^2=\left(\sqrt{2t-5}\right)^2$ (squaring of each side) $\Rightarrow \quad 2t+6=2t-5$

$2t-2t=-6-5 \quad \Rightarrow \quad 0\neq-11$

Hence the solution set $=\phi$

(viii) $\sqrt{\frac{x+1}{2x+5}}=2,\ x\neq-\frac{5}{2}$

Solution: $\sqrt{\frac{x+1}{2x+5}}=2 \quad \Rightarrow \quad \left(\sqrt{\frac{x+1}{2x+5}}\right)^2=(2)^2$ (squaring of each side)

$\frac{x+1}{2x+5}=4 \quad \Rightarrow \quad 4(2x+5)=x+1$

$8x+20=x+1 \quad \Rightarrow \quad 8x-x=1-20$

$7x=-19 \quad \Rightarrow \quad x=-\frac{19}{7}$

Check: Substituting $x=-\frac{19}{7}$ in the original equation, we get

$\sqrt{\frac{-\frac{19}{7}+1}{2\left(-\frac{19}{7}\right)+5}}=2 \quad \Rightarrow \quad \sqrt{\frac{-\frac{12}{7}}{-\frac{3}{7}}}=2$

$\sqrt{\frac{12}{7}\times\frac{7}{3}}=2 \quad \Rightarrow \quad \sqrt{4}=2$

$2=2$ (which is true)

Since $x=-\frac{19}{7}$ makes the statement true, therefore the solution is correct.

Hence the solution set $=\{-\frac{19}{7}\}$

Equation Involving Absolute Value:

Another type of linear equation is the one that contains absolute value. To solve equation involving absolute value we first give the following definition.

Definition:

The absolute value of a real number 'a' denoted by $|a|$, is defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

e.g. $|6| = 6$, $|0| = 0$ and $|-6| = -(-6) = 6$

Some properties of Absolute Value:

If $a, b \in R$, then

(i) $|a| \geq 0$ (ii) $|-a| = |a|$

(iii) $|ab| = |a|.|b|$ (iv) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0.$

Solving Linear Equations Involving Absolute Value:

Keeping in mind the definition of absolute value we can immediately say that:

$|x| = 3$ is equivalent to $x = 3$ or $x = -3$,

Because $x = +3$ or $x = -3$ make $|x| = 3$ a true statement.

For solving an equation involving absolute value, we express the given equation as an equivalent compound sentence and solve each part separately.

Example: Solve and check, $|2x + 3| = 11$

Solution: By definition, depending on whether $(2x + 3)$ is positive or negative the given equation is equivalent to:

$+(2x + 3) = 11$ or $-(2x + 3) = 11$

In practice, these two equations are usually written as

$2x + 3 = +11$ or $2x + 3 = -11$

$2x = 8$ or $2x = -14$

$x = 4$ or $x = -7$

Check: Substituting $x = 4$, in the original equation, we get

$|2(4) + 3| = 11 \Rightarrow 11 = 11$ (which is true)

New substituting $x = -7$, we have

$|2(-7) + 3| = 11 \Rightarrow |-11| = 11 \Rightarrow 11 = 11$ (which is true)

Hence $x = 4$, $x = -7$ are the solutions to the given equation.

Hence, solution set $= \{-7, 4\}$

Example: Solve $|8x - 3| = |4x + 5|$

Solution: Since two numbers having the same absolute value are either equal or differ in sign, therefore, the given equation is equivalent to

$8x - 3 = 4x + 5$ or $8x - 3 = -(4x + 5)$

$4x = 8$ or $12x = -2$

$x = 2$ or $x = -\frac{1}{6}$

On checking we find that $x = 2$, $x = -\frac{1}{6}$ both satisfy the original equation.

Hence, solution set $= \left\{-\frac{1}{6}, 2\right\}$

Example: Solve and check $|3x + 10| = 5x + 6$

Solution: The given equation is equivalent to

$\pm(3x+10) = 5x+6$

i.e., $3x + 10 = 5x + 6$ or $3x + 10 = -(5x + 6)$

$-2x = -4$ or $8x = -16$

$x = 2$ or $x = -2$

On checking in the original equation we see that $x = -2$ does not satisfy it.
Hence the only solution is $x = 2$.

Solved Exercise 7.2

1. Identify the following statements as True or False.

(i) $|x| = 0$ has only one solution.

(ii) All absolute value equations have two solutions.

(iii) The equation $|x| = 2$ is equivalent to $x = 2$ or $x = -2$.

(iv) The equation $|x - 4| = -4$ has no solution.

(v) The equation $|2x - 3| = 5$ is equivalent to $2x - 3 = 5$ or $2x + 3 = 5$.

Solution: **(i)** T **(ii)** F **(iii)** T **(iv)** T **(v)** F

2. Solve for x

(i) $|3x - 5| = 4$

Solution: $|3x - 5| = 4$

By definition, the given equation is equivalent to

$(3x-5) = +4$ or $(3x - 5) = -4$

$3x - 5 = 4$ or $3x - 5 = -4$

$3x = 9$

$x = 3$ or $x = \frac{1}{3}$

Check: Substituting $x = 3$ in the original equation, we get

$|3(3) - 5| = 4 \Rightarrow |9 - 5| = 4 \Rightarrow 4 = 4$ (which is true)

Substituting $x = \frac{1}{3}$ in the original equation, we get

$\left|3\left(\frac{1}{3}\right) - 5\right| = 4 \Rightarrow |1 - 5| = 4 \Rightarrow |-4| = 4$

$4 = 4$ (which is true)

Hence, solution set $= \{3, \frac{1}{3}\}$

(ii) $\frac{1}{2}|3x+2|-4=11$

Solution: $\frac{1}{2}|3x+2|-4=11$

$$\frac{1}{2}|3x+2|=11+4$$

$$\frac{1}{2}|3x+2|=15$$

$$|3x+2|=30$$

By definition, the given equation is equivalent to

$3x+2=+30$ or $3x+2=-30$

$3x=30-2$ $3x=-30-2$

$3x=28$ $3x=-32$

$x=\frac{28}{3}$ $x=-\frac{32}{3}$

Check: Substituting $x=\frac{28}{3}$ in the original equation, we get

$$\frac{1}{2}\left|3\left(\frac{28}{3}\right)+2\right|-4=11 \Rightarrow \frac{1}{2}|28+2|-4=11$$

$$\frac{1}{2}|30|-4=11 \Rightarrow \frac{1}{2}|30|-4=11$$

$$15-4=11 \Rightarrow 11=11 \quad \text{(which is true)}$$

Substituting $x=-\frac{32}{3}$ in the original equation, we get

$$\frac{1}{2}\left|3\left(-\frac{32}{3}\right)+2\right|-4=11 \Rightarrow \frac{1}{2}|-32+2|-4=11$$

$$\frac{1}{2}|-30|-4=11 \Rightarrow \frac{1}{2}|30|-4=11$$

$$15-4=11 \Rightarrow 11=11 \quad \text{(which is true)}$$

Hence, solution set $=\left\{\frac{28}{3},-\frac{32}{3}\right\}$

(iii) $|2x+5|=11$

Solution: $|2x+5|=11$

By definition, the given equation is equivalent to

$2x+5=+11$ or $2x+5=-11$

$2x=11-5$ $2x=-11-5$

$2x=6$ $2x=-16$

$x=3$ $x=-8$

Check: Substituting $x=3$ in the original equation, we get

$|2(3)+5| = 11 \quad \Rightarrow \quad |6+5| = 11$

$|11| = 11 \quad \Rightarrow \quad 11 = 11$ (which is true)

Substituting $x = -8$ in the original equation, we get

$|2(-8)+5| = 11 \quad \Rightarrow \quad |-16+5| = 11$

$|-11| = 11 \quad \Rightarrow \quad 11 = 11$ (which is true)

Hence, solution set = $\{-8, 3\}$

(iv) $|3+2x| = |6x-7|$

Solution: $|3+2x| = |6x-7|$

By definition, the given equation is equivalent to

$(3+2x) = +(6x-7)$ or $(3+2x) = -(6x-7)$

$3+2x = 6x-7$ $\qquad$ $3+2x = -6x+7$

$2x-6x = -7-3$ $\qquad$ $2x+6x = -3+7$

$-4x = -10$ $\qquad$ $8x = 4$

$x = \frac{10}{4}$ $\qquad$ $x = \frac{1}{2}$

$x = \frac{5}{2}$

Check: Substituting $x = \frac{5}{2}$ in the original equation, we get

$$\left|3+2\left(\frac{5}{2}\right)\right| = \left|6\left(\frac{5}{2}\right)-7\right| \Rightarrow |3+5| = |15-7|$$

$|8| = |8| \quad \Rightarrow \quad 8 = 8$ (which is true)

Substituting $x = \frac{1}{2}$ in the original equation, we get

$$\left|3+2\left(\frac{1}{2}\right)\right| = \left|6\left(\frac{1}{2}\right)-7\right| \Rightarrow |3+1| = |3-7|$$

$|4| = |-4| \quad \Rightarrow \quad 4 = 4$ (which is true)

Hence, solution set = $\left\{\frac{5}{2}, \frac{1}{2}\right\}$

(v) $|x+2| - 3 = 5 - |x+2|$

Solution: $|x+2| - 3 = 5 - |x+2|$

$|x+2| + |x+2| = 5+3$

$2|x+2| = 8$

$\Rightarrow \quad |x+2| = 4$

By definition, the given equation is equivalent to

$x+2 = +4$ or $x+2 = -4$

$x = 4-2$ $\qquad$ $x = -4-2$

$x = 2 \qquad\qquad x = -6$

Check: Substituting $x = 2$ in the original equation, we get

$|2+2| - 3 = 5 - |2+2| \Rightarrow |4| - 3 = 5 - |4|$

$4 - 3 = 5 - 4 \Rightarrow 1 = 1$ (which is true)

Substituting $x = -6$ in the original equation, we get

$|-6+2| - 3 = 5 - |-6+2| \Rightarrow |-4| - 3 = 5 - |-4|$

$4 - 3 = 5 - 4 \Rightarrow 1 = 1$ (which is true)

Hence, solution set $= \{2, -6\}$

(vi) $\frac{1}{2}|x+3| + 21 = 9$

Solution: $\frac{1}{2}|x+3| + 21 = 9$

$\frac{1}{2}|x+3| = 9 - 21 \Rightarrow \frac{1}{2}|x+3| = -12$

This is not possible. Hence solution set $= \phi$

(vii) $\left|\frac{3-5x}{4}\right| - \frac{1}{3} = \frac{2}{3}$

Solution: $\left|\frac{3-5x}{4}\right| - \frac{1}{3} = \frac{2}{3}$

$\left|\frac{3-5x}{4}\right| = \frac{2}{3} + \frac{1}{3} \Rightarrow \left|\frac{3-5x}{4}\right| = 1$

By definition, the given equation is equivalent to

$\left|\frac{3-5x}{4}\right| = +1 \qquad$ or $\qquad \left|\frac{3-5x}{4}\right| = -1$

$3 - 5x = 4 \qquad\qquad 3 - 5x = -4$

$-5x = 4 - 3 \qquad\qquad -5x = -3 - 4$

$-5x = 1 \qquad\qquad -5x = -7$

$x = -\frac{1}{5} \qquad\qquad x = \frac{7}{5}$

Check: Substituting $x = \frac{1}{5}$ in the original equation, we get

$\left|\frac{3 - 5\left(-\frac{1}{5}\right)}{4}\right| - \frac{1}{3} = \frac{2}{3} \Rightarrow \left|\frac{3+1}{4}\right| - \frac{1}{3} = \frac{2}{3}$

$\left|\frac{4}{4}\right| - \frac{1}{3} = \frac{2}{3} \Rightarrow 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow \frac{2}{3} = \frac{2}{3}$ (which is true)

Substituting $x = \frac{7}{5}$ in the original equation, we get

$$\left|\frac{3-5\left(\frac{7}{5}\right)}{4}\right|-\frac{1}{3}=\frac{2}{3} \quad \Rightarrow \quad \left|\frac{3-7}{4}\right|-\frac{1}{3}=\frac{2}{3}$$

$$\left|-\frac{4}{4}\right|-\frac{1}{3}=\frac{2}{3} \quad \Rightarrow \quad 1-\frac{1}{3}=\frac{2}{3} \quad \Rightarrow \quad \frac{2}{3}=\frac{2}{3} \quad \text{(which is true)}$$

Hence solution set = $\left\{-\frac{1}{5},\frac{7}{5}\right\}$

(viii) $\left|\frac{x+5}{2-x}\right|=6$

Solution: $\left|\frac{x+5}{2-x}\right|=6$

By definition, the given equation is equivalent to

$\frac{x+5}{2-x}=+6$ or $\frac{x+5}{2-x}=-6$

$x+5=6(2-x)$	$x+5=-6(2-x)$
$x+5=12-6x$	$x+5=-12+6x$
$x+6x=12-5$	$x-6x=-12-5$
$7x=7$	$-5x=-17$
$x=1$	$x=\frac{17}{5}$

Check: Substituting x = 1 in the original equation, we get

$$\left|\frac{1+5}{2-1}\right|=6 \quad \Rightarrow \quad \left|\frac{6}{1}\right|=6 \quad \Rightarrow \quad 6=6 \quad \text{(which is true)}$$

Substituting $x=\frac{17}{5}$ in the original equation, we get

$$\left|\frac{\frac{17}{5}+5}{2-\frac{17}{5}}\right|=6 \Rightarrow \left|\frac{\frac{42}{5}}{-\frac{7}{5}}\right|=6 \quad \Rightarrow \quad \left|\frac{42}{5}\times\left(-\frac{5}{7}\right)\right|=6$$

$$|-6|=6 \Rightarrow \quad 6=6 \quad \text{(which is true)}$$

Hence, solution set = $\left\{1,\frac{17}{5}\right\}$

LINEAR INEQUALITIES:

We discussed an important comparing property called ordering real numbers stating that one number is less than or greater than another number. This order relation helps us to compare two real numbers 'a' and 'b' when $a \neq b$. This

comparability is of primary importance in many applications. We may compare prices, heights, weights, temperatures, distances, costs of manufacturing, distances, time etc. The inequality symbols '<' and '>' for ≠ were introduced by an English mathematician Thomas Harriot (1560 – 1621).

Defining Inequalities:

Let a, b be real numbers. Then 'a' is greater than 'b' if the difference $a - b$ is positive and we denote this order relation by the inequality $a > b$. An equivalent statement is that b is less than a, symbolized by $b < a$. Similarly, if $a - b$ is negative, then 'a' is less than 'b' and expressed in symbols as $a < b$.

Sometimes we know that one number is either less than another number or equal to it. But we do not know which one is the case. In such a situation we use the symbol "≤" which is read as "less than or equal to". Likewise, the symbol "≥" is used to mean "greater than or equal to". The symbols <, >, ≤ and ≥ are also called **inequality signs**. The inequalities $x > y$ and $y < x$ are known as strict (or strong) whereas the inequalities $x \geq y$ and $y \leq x$ are called non- strict (or weak).

If we combine $a < b$ and $b < c$ we get a double inequality written in a compact form as $a < b < c$ which means "b lies between a and c" and read as "a is less than b less than c". Similarly, "$a \leq b \leq c$" is read as "b is between a and c, inclusive."

A linear inequality in one variable x s an inequality in which the variable x occurs only to the first power and is of the form.

$$ax + b < 0, a \neq 0$$

Where a and b are real numbers. We may replace the symbol < by >, ≤ or ≥ also.

Properties of Inequalities:

The properties of inequalities which we are going to use in solving inequalities in one variable are as under.

1. Law of Trichotomy:

For any $a, b \in R$, one and only one of the following statements is true.

$$a < b \text{ or } a = b, \text{ or } a > b$$

An important special case of this property is the case for $b = 0$; namely,
$a < 0$ or $a = 0$ or $a > 0$ for any $a \in R$.

2. Transitive property:

Let $a, b, c \in R$.

(i) If $a > b$ and $b > c$, then $a > c$ (ii) If $a < b$ and $b < c$, then $a < c$.

3. Additive Closure Property:

For $a, b, c \in R$,

(i) If $a > 0$ and $b > 0$, then $a + b > 0$ If $a < 0$ and $b < 0$, then $a + b < 0$

(ii) If $a > b$, then $a + c > b + c$ If $a < b$, then $a + c < b + c$.

4. Multiplicative Property:

Let $a, b, c, d \in R$.

(i) If $a > 0$ and $b > 0$, then $ab > 0$ whereas $a < 0$ and $b < 0 \Rightarrow ab > 0$.

(ii) If $a > b$ and $c > 0$, then $ac > bc$ or if $a < b$ and $c > 0$, then $ac < bc$.

(iii) If $a > b$ and $c < 0$, then $ac < bc$. or if $a < b$ and $c < 0$, then $ac > bc$.

The above property (iii) states that the sign of inequality is reversed if each side is multiplied by a negative real number.

(iv) If $a > b$ and $c > d$, then $ac > bd$.

Solving Inequalities in one Variable:

The method of solving an algebraic inequality in one variable is explained with the help of following examples.

Example-1: Solve $9 - 7x > 19 - 2x$, where $x \in R$.

Solution: $9 - 7x > 19 - 2x$

$$9 - 5x > 19$$
$$-5x > 10$$
$$x < -2$$

Hence, solution set $= \{x \mid x < -2\}$

Example-2: Solve $\frac{1}{2}x - \frac{2}{3} \le x + \frac{1}{3}$, where $x \in R$.

Solution: $\frac{1}{2}x - \frac{2}{3} \le x + \frac{1}{3}$

To clear fractions we multiply each side by 6, the L.C.M. of 2 and 3 and get.

$$6\left[\frac{1}{2}x - \frac{2}{3}\right] \le 6\left[x + \frac{1}{3}\right]$$

or $3x - 4 \le 6x + 2$

or $3x \le 6x + 6$

or $-3x \le 6$

or $x \ge -2$

Hence, solution set $= \{x \mid x \ge -2\}$.

Example-3: Solve the double inequality $-2 < \frac{1-2x}{3} < 1$, where $x \in R$.

Solution: The given inequality is a double inequality and represents two separate inequalities.

$$-2 < \frac{1-2x}{3} \quad \text{and} \quad \frac{1-2x}{3} < 1$$

$$-2 < \frac{1-2x}{3} < 1$$
$$-6 < 1 - 2x < 3$$
$$-7 < -2x < 2$$
$$\frac{7}{2} > \quad x > -1$$

i.e., $-1 < x < 3.5$

Hence, solution set = $\{x \mid -1 < x < 3.5\}$.

Example-4: Solve the inequality $4x - 1 \le 3 \le 7 + 2x$, where $x \in R$.

Solution: The given inequality holds if and only if both the separate inequalities $4x - 1 \le 3$ and $3 \le 7 + 2x$ hold. We solve each of these inequalities separately.

$$4x - 1 \le 3$$
$$4x \le 3 + 1$$
$$4x \le 4$$
$$x \le 1 \quad \ldots\ldots (i)$$

Now
$$3 \le 7 + 2x$$
$$3 - 7 \le 2x$$
$$-4 \le 2x$$
$$-2 \le x$$
$$x \ge -2 \quad \ldots\ldots (ii)$$

Combining (i) and (ii), we have

$$-2 \le x \le 1$$

Hence, solution set = $\{x \mid -2 \le x \le 1\}$.

Solved Exercise 7.3

1. Solve the following inequalities.

(i) $3x + 1 < 5x - 4$

Solution: $3x + 1 < 5x - 4 \Rightarrow 3x - 5x < -4 - 1$

$-2x < -5 \Rightarrow 2x > 5$

$\Rightarrow x > \frac{5}{2}$

Hence, solution set = $\{x \mid x > \frac{5}{2}\}$

(ii) $4x - 10.3 \le 21x - 1.8$

Solution:
$$4x - 10.3 \le 21x - 1.8$$
$$4x - 21x \le 10.3 - 1.8$$
$$-17x \le 8.5$$
$$x \ge -0.5$$

Hence, solution set = $\{x \mid x \ge -0.5\}$

(iii) $4 - \frac{1}{2}x \ge -7 + \frac{1}{4}x$

Solution: $4 - \frac{1}{2}x \ge -7 + \frac{1}{4}x \Rightarrow -\frac{1}{2}x - \frac{1}{4}x \ge -7 - 4$

$\left(-\frac{1}{2} - \frac{1}{4}\right)x \ge -11 \Rightarrow -\frac{3}{4}x \ge -11$

$x \le 11 \times \frac{4}{3} \quad \Rightarrow \quad x \le \frac{44}{3}$

Hence, solution set = $\{x \mid x \le \frac{44}{3}\}$

(iv) $x - 2(5 - 2x) \ge 6x - 3\frac{1}{2}$

Solution: $x - 2(5 - 2x) \ge 6x - 3\frac{1}{2} \quad \Rightarrow \quad x - 10 + 4x \ge 6x - \frac{7}{2}$

$5x - 10 \ge 6x - \frac{7}{2} \quad \Rightarrow \quad 5x - 6x \ge 10 - \frac{7}{2}$

$-x \ge \frac{13}{2} \quad \Rightarrow \quad x \le -6.5$

Hence, solution set = $\{x \mid x \le -6.5\}$

(v) $\frac{3x+2}{9} - \frac{2x+1}{3} > -1$

Solution: $\frac{3x+2}{9} - \frac{2x+1}{3} > -1$

$9 \times \frac{3x+2}{9} - 9 \times \frac{2x+1}{3} > -9$

$(3x+2) - (6x+3) > -9$

$3x + 2 - 6x - 3 > -9$

$-3x - 1 > -9$

$-3x > -9 + 1$

$-3x > -8$

$x < \frac{8}{3}$

Hence, solution set = $\{x \mid x < \frac{8}{3}\}$

(vi) $3(2x+1) - 2(2x+5) < 5(3x - 2)$

Solution: $3(2x+1) - 2(2x+5) < 5(3x - 2)$

$6x + 3 - 4x - 10 < 15x - 10$

$2x - 7 < 15x - 10$

$2x - 15x < -10 + 7$

$-13x < -3$

$x > \frac{3}{13}$

Hence, solution set = $\{x \mid x > \frac{3}{13}\}$

(vii) $3(x-1) - (x-2) > -2(x+4)$

Solution: $3(x-1) - (x-2) > -2(x+4)$

$3x - 3 - x + 2 > -2x - 8$

$$2x - 1 > -2x - 8$$
$$2x + 2x > 1 - 8$$
$$4x > -7$$
$$x > -\frac{7}{4}$$

Hence, solution set = $\{x \mid x > -\frac{7}{4}\}$

(viii) $2\frac{2}{3}x+\frac{2}{3}(5x-4)>-\frac{1}{3}(8x+7)$

Solution: $2\frac{2}{3}x + \frac{2}{3}(5x-4) > \frac{-1}{3}(8x+7)$

$$\frac{8}{3}x + \frac{2}{3}(5x-4) > -\frac{1}{3}(8x+7)$$
$$3\times\frac{8}{3}x + 3\times\frac{2}{3}(5x-4) > -\frac{1}{3}(8x+7)\times 3$$
$$8x + 2(5x-4) > -(8x+7)$$
$$8x + 10x - 8 > -8x - 7$$
$$8x + 8x + 10x > -7 + 8$$
$$26x > 1$$
$$x > \frac{1}{26}$$

Hence, solution set = $\left(x \quad x > \frac{1}{26}\right)$

2. Solve the following inequalities.

(i) $-4 < 3x + 5 < 8$

Solution: $-4 < 3x + 5 < 8$

$$-4 - 5 < 3x < 8 - 5$$
$$-9 < 3x < 3$$
$$-3 < x < 1$$

Hence, solution set = $\{x \mid -3 < x < 1\}$

(ii) $-5 \le \frac{4-3x}{2} < 1$

Solution: $-5 \le \frac{4-3x}{2} < 1$

$$-10 \le 4 - 3x < 2$$
$$-10 - 4 \le -3x < 2 - 4$$
$$-14 \le -3x < -2$$
$$-\frac{14}{3} \le -3\frac{3x}{3} < -\frac{2}{3}$$
$$\Rightarrow \quad \frac{14}{3} \ge x > \frac{2}{3}$$

Or $\frac{2}{3} < x \le \frac{14}{3}$

Hence, solution set = $\{x \mid \frac{2}{3} < x \le \frac{14}{3}\}$

(iii) $-6 < \frac{x-2}{4} < 6$

Solution: $-6 < \frac{x-2}{4} < 6$

$$-24 < x - 2 < 24$$
$$-24 + 2 < x < 24 + 2$$
$$-22 < x < 26$$

Hence, solution set = $\{x \mid -22 < x < 26\}$

(iv) $3 \ge \frac{7-x}{2} \ge 1$

Solution: $3 \ge \frac{7-x}{2} \ge 1$

$$6 \ge 7 - x \ge 2$$
$$6 - 7 \ge -x \ge 2 - 7$$
$$-1 \ge -x \ge -5$$
$$\Rightarrow 1 \le x \le 5$$

Hence, solution set = $\{x \mid 1 \le x \le 5\}$

(v) $3x - 10 \le 5 < x + 3$

Solution: $3x - 10 \le 5 < x + 3$

$\Rightarrow 3x - 10 \le 5$ or $5 < x + 3$

$3x \le 5 + 10$ or $x + 3 \ge 5$

$3x \le 15$ $\quad x < 5 - 3$

$x \le 5$ ...... (i) $\quad x < 2$ .... (ii)

Combining (i) and (ii), we have

$$2 \le x \le 5$$

Hence, solution set = $\{x \mid 2 < x \le 5\}$

(vi) $-3 \le \frac{x-4}{-5} < 4$

Solution: $-3 \le \frac{x-4}{-5} < 4$

$$-15 \le -(x-4) < 20$$
$$-15 \le -x + 4 < 20$$
$$-15 - 4 \le -x < 20 - 4$$
$$-19 \le -x < 16$$

or $19 \ge x > -16$

or $-16 < x \le 19$

Hence, solution set = $\{x \mid -16 < x \leq 19\}$

(vii) $1 - 2x < 5 - x \leq 25 - 6x$

Solution: $1 - 2x < 5 - x \leq 25 - 6x$

$\Rightarrow 1 - 2x < 5 - x$ or $5 - x \leq 25 - 6x$

$-2x + x < 5 - 1$ $\quad$ $-x + 6x \leq 25 - 5$

$-x < 4$ $\quad$ $5x \leq 20$

$x > -4$ …… (i) $\quad$ $x \leq 4$ …… (ii)

Combining (i) and (ii), we have

$-4 < x \leq 4$

Hence, solution set = $\{x \mid -4 < x \leq 4\}$

(viii) $3x - 2 < 2x + 1 < 4x + 17$

Solution: $3x - 2 < 2x + 1 < 4x + 17$

$\Rightarrow 3x - 2 < 2x + 1$ or $2x+1 < 4x+17$

$3x - 2x < 2 + 1$ $\quad$ $2x - 4x < 17 - 1$

$x < 3$ …. (i) $\quad$ $-2x < 16$

$x > -8$ …. (ii)

Combining (i) and (ii), we have

$-8 < x < 3$

Hence, solution set = $\{x \mid -8 < x < 3\}$

Solved Review Exercise 7

1. Choose the correct answer.

(i) Which of the following is the solution of the inequality $3 - 4x \leq 11$?.....

(a) -8 (b) -2 (c) $-\frac{14}{4}$ (d) None of these

(ii) A statement involving any of the symbols <, > ≤ or ≥ is called…..

(a) equation (b) identity (c) inequality (d) linear equation

(iii) x = …… is a solution of the inequality $-2 < x < \frac{3}{2}$.

(a) -5 (b) 3 (c) 0 (d) $\frac{3}{2}$

(iv) If x is no larger than 10, then….

(a) $x \geq 8$ (b) $x \leq 10$ (c) $x < 10$ (d) $x > 10$

(v) If the capacity c of an elevator is at most 1600 pounds, then….

(a) $c < 1600$ (b) $c \geq 1600$ (c) $c \leq 1600$ (d) $c > 1600$

(vi) x = 0 is a solution of the inequality……

(a) $x > 0$ (b) $3x + 5 < 0$ (c) $x + 2 < 0$ (d) $x - 2 < 0$

Solution: (i) b (ii) c (iii) c (iv) b (v) c (vi) d

2. Identify the following statements as True or False.

(i) The equation $3x - 5 = 7 - x$ is a linear equation. ______

(ii) The equation $x - 0.3x = 0.7x$ is an identity. _____

(iii) The equation $-2x + 3 = 8$ is equivalent to $-2x = 11$. _____

(iv) To eliminate fractions, we multiply each side of an equation by the L.C.M. of denominators. _____

(v) $4(x + 3) = x + 3$ is a conditional equation. _____

(vi) The equation $2(3x + 5) = 6x + 12$ is an inconsistent equation. _____

(vii) To solve $\frac{2}{3}x = 12$, we should multiply each side by $\frac{2}{3}$. _____

(viii) Equations having exactly the same solution are called equivalent equations. _____

(ix) A solution that does not satisfy the original equation is called extraneous solution. _____

Solution: (i) T (ii) T (iii) F (iv) T (v) T
(vi) T (vii) F (viii) T (ix) T

3. Answer the following short questions.

(i) Define a linear inequality in one variable.

Solution: Linear inequality in one variable:

A linear inequality in one variable x is an inequality in which the variable x occurs only to the first power and is of the form. $ax + b < 0$

Where 'a' and 'b' are real numbers: we may replace the symbol '<' by '>' , '≤' or '≥', etc.

(ii) State the trichotomy and transitive properties of inequalities.

Solution: Law of trichotomy of inequalities:

For any $a, b \in R$, one and only one of the following statements is true.
$a < b$ or $a = b$ or $a > b$.

Transitive property of inequalities:

Let $a, b, c \in R$.
(i) If $a > b$ and $b > c$, then $a > c$.
(ii) If $a < b$ and $b > c$, then $a < c$.

(iii) The formula relating degrees Fahrenheit to degrees Celsius is $F = \frac{9}{5}C + 32$. For what value of C if $F < 0$?

Solution: $F = \frac{9}{5}C + 32$

Given $F < 0$

Subtracting 32 both sides, we get

$$\frac{9}{5}C + 32 - 32 < 0 - 32$$

$$\frac{9}{5}C < -32$$

Multiplying $\frac{5}{9}$ both sides, we get

$$\frac{5}{9}\times\frac{9}{5}C < \frac{5}{9}\times(-32)$$
$$C<\frac{-160}{9}$$

(iv) Seven times the sum of an integer and 12 is at least 50 and at most 60. Write and solve the inequality that expresses this relationship.

Solution: $7(x+12)\geq 50$ and $7(x+12)\leq 60$

$$7x+84\geq 50 \qquad 7x+84\leq 60$$
$$7x\geq 50-84 \qquad 7x\leq 60-84$$
$$7x\geq -34 \qquad 7x\leq -24$$
$$x\geq -\frac{34}{7} \qquad x\leq -\frac{24}{7}$$

4. Solve each of the following and check for extraneous solution, if any.

(i) $\sqrt{2t+4}=\sqrt{t-1}$

Sol: $\sqrt{2t+4}=\sqrt{t-1} \Rightarrow \left(\sqrt{2t+4}\right)^2=\left(\sqrt{t-1}\right)^2$

$$2t+4=t-1 \Rightarrow 2t-t=-1-4 \Rightarrow t=-5$$

Check: Substituting $t=-5$ in the original equation, we get

$$\sqrt{2(-5)+4}=\sqrt{(-5)-1} \Rightarrow \sqrt{-10+4}=\sqrt{-6}$$
$$\sqrt{-6}=\sqrt{-6} \quad \text{(which is not true)}$$

Since, $t=-5$ does not makes the original statement true, therefore the solution is not correct.

Hence, solution set $=\Phi$

(ii) $\sqrt{3x-1}-2\sqrt{8-2x}=0$

Solution:

$$\sqrt{3x-1}-2\sqrt{8-2x}=0 \Rightarrow \sqrt{3x-1}=2\sqrt{8-2x}$$
$$\left(\sqrt{3x-1}\right)^2=\left(2\sqrt{8-2x}\right)^2 \Rightarrow 3x-1=4(8-2x)$$
$$3x-1=32-8x \Rightarrow 3x+8x=32+1$$
$$11x=33 \Rightarrow x=3$$

Check: Substituting $x=3$ in the original equation, we get

$$\sqrt{3(3)-1}-2\sqrt{8-2(3)}=0 \Rightarrow \sqrt{9-1}-2\sqrt{8-6}=0$$
$$\sqrt{8}-2\sqrt{2}=0 \Rightarrow 2\sqrt{2}-2\sqrt{2}=0$$
$$0=0 \quad \text{(which is true)}$$

Since, $x=3$ makes the original statement true, therefore the solution is correct.

Hence, solution set $=\{3\}$

5. Solve for x

(i) $|3x+14|-2=5x$

Solution: $|3x+14|-2=5x$

$$|3x+14|=5x+2$$

By definition, the given equation is equivalent to

$3x+14=+(5x+2)$ or $3x+14=-(5x+2)$

$3x+14=+5x+2$ or $3x+14=-5x-2$

$3x-5x=2-14$ or $3x+5x=-2-14$

$-2x=-12$ or $8x=-16$

$x=6$ or $x=-2$

Check: Substituting $x=-2$ in the original equation, we get

$|3(-2)+14|-2=5(-2) \Rightarrow |-6+14|-2=-10$

$|8|-2=-10 \Rightarrow 8-2=-10$

$6\neq-10$ (which is not true)

Since, $x=-2$ does not makes the original statement true, therefore the solution is not correct.

Substituting $x=6$ in the original equation, we get

$|3(6)+14|-2=5(6) \Rightarrow |18+14|-2=30$

$|32|-2=30 \Rightarrow 32-2=30$

$30=30$ (which is true)

Since, $x=6$ makes the original statement true, therefore the solution is correct.

Hence, solution set $=\{6\}$

(ii) $\frac{1}{3}|x-3|=\frac{1}{2}|x+2|$

Solution: $\frac{1}{3}|x-3|=\frac{1}{2}|x+2|$

By definition, the given equation is equivalent to

$\frac{1}{3}(x-3)=+\frac{1}{2}(x+2)$ or $\frac{1}{3}(x-3)=-\frac{1}{2}(x+2)$

$6\times\frac{1}{3}(x-3)=6\times\frac{1}{2}(x+2)$ $6\times\frac{1}{3}(x-3)=-\frac{1}{2}(x+2)\times6$

$2(x-3)=3(x+2)$ $2(x-3)=-3(x+2)$

$2x-6=3x+6$ $2x-6=-3x-6$

$2x-3x=6+6$ $2x+3x=-6+6$

$-x=12$ $5x=0$

$x=-12$ $x=0$

Check: Substituting $x=0$ in the original equation, we get

$\frac{1}{3}|0-3|=\frac{1}{2}|0+2| \Rightarrow \frac{1}{3}|-3|=\frac{1}{2}|2|$

$$\frac{1}{3}(-3)=\frac{1}{2}(2) \quad \Rightarrow \quad 1=1 \quad \text{(which is true)}$$

Since, x = 0 makes the original statement true, therefore the solution is correct.
Substituting x = –12 in the original equation, we get

$$\frac{1}{3}|-12-3|=\frac{1}{2}|-12+2| \quad \Rightarrow \quad \frac{1}{3}|-15|=\frac{1}{2}|-10|$$

$$\frac{1}{3}(15)=\frac{1}{2}(10) \quad \Rightarrow \quad 5=5 \quad \text{(which is true)}$$

Since, x = – 12 makes the original statement true, therefore the solution is correct.
Hence, solution set = {– 12, 0}

6. Solve the following inequality.

(i) $-\frac{1}{3}x+5\leq 1$

Solution:

$$-\frac{1}{3}x+5\leq 1 \quad \Rightarrow \quad -\frac{1}{3}x\leq 1-5$$

$$-\frac{1}{3}x\leq -4 \quad \Rightarrow \quad -x\leq -12$$

$$x\geq 12$$

(ii) $-3<\frac{1-2x}{5}<1$

Solution:

$$-3<\frac{1-2x}{5}<1 \quad \Rightarrow \quad -15<1-2x<5$$

$$-15-1<-2x<5-1 \quad \Rightarrow \quad -16<-2x<4$$

$$-\frac{16}{2}<-\frac{2x}{2}<\frac{4}{2} \quad \Rightarrow \quad -8<-x<2$$

$$8>x>-2$$

SUMMARY

- Linear equation in one variable x is ax + b = 0 where a, b $\in$ R, a $\neq$ 0.
- Solution to an equation is that value of x which makes it a true statement.
- Two linear equations are called equivalent if they have exactly the same solution.
- An inconsistent equation is that whose solution set is ϕ.
- Additive property of equality:
  If a = b, then a + c = b + c and a – c = b – c. $\forall$ a, b, c $\in$ R
- Multiplicative property of equality: If a = b, then ac = bc.

- Cancellation property: If $a + c = b + c$, then $a = b$.
  If $ac = bc$, $c \neq 0$ then $a = b$, $\forall$ a, b, c $\in R$
- To solve an equation we find a sequence of equivalent equations to isolate the variable x on one side of the equality to get solution.
- A radical equation is that in which the variable occurs under the radical. It must be checked for any extraneous solution (s).
- Absolute value of a real number a is defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

- Properties of Absolute value:
  If a, b $\in$ R, then
  - **(i)** $|a| \geq 0$
  - **(ii)** $|-a| = |a|$
  - **(iii)** $|ab| = |a| \cdot |b|$
  - **(iv)** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$
  - **(v)** $|x| = a$ is equivalent to $x = a$ or $x = -a$.
- Inequality symbols are $<$, $>$, $\leq$, $\geq$
- A linear inequality in one variable x is $ax + b < 0$, $a \neq 0$.
- Properties of Inequality:
  - (a) *Law of Trichotomy*
    If a, b $\in$ R then $a < b$ or $a = b$ or $a > b$
  - (b) Transitive law If $a > b$ and $b > c$, then $a > c$.
  - (c) Multiplication and division:
    - **(i)** If $a > b$, $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.
    - **(ii)** If $a > b$, $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

UNIT 8

LINEAR GRAPHS & THEIR APPLICATION

Unit Outlines

8.1 *Introduction* 8.2 *Cartesian plane*
8.3 *Conversion Graphs*
8.4 *Graphical Solution of Equations in two variables*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- identify pair of real numbers as an ordered pair.
- recognize an ordered pair through different examples
- describe rectangular or Cartesian plane consisting of two number lines interesting at right angles at the point O.
- identify origin (0) and coordinate axes (horizontal and vertical axes or x-axis and y-axis) in the rectangular plane.
- locate an ordered pair (a, b) as a point in the rectangular plane and recognize.
  - 'a' as the x-coordinate (or abscissa)
  - 'b' as the y-coordinate (or ordinate)
- draw different geometrical shapes (e.g., line segment, triangle and rectangle etc.) by joining a set of given points.
- construct a table for pairs of values satisfying a linear equation in two variables.
- plot the pairs of points to obtain the graph of a given expression.
- choose an appropriate scale to draw a graph.
- draw a graph of
  - an equation of the form $y = c$
  - an equation of the form $x = a$
  - an equation of the form $y = mx$
  - an equation of the form $y = mx + c$
- draw a graph from a given table of (discrete) values.
- solve appropriate real life problems.
- interpret conversion graph as a linear graph relating to two quantities which are in direct proportion.
- read a given graph to know one quantity corresponding to another.
- read the graph for conversions of the form.
  - miles and kilometers

- ★ acres and hectares
- ★ degrees Celsius and degrees Fahrenheit
- ★ Pakistani currency and another currency, etc
- ✽ Solve simultaneous linear equations in two variables using graphical method

CARTESIAN PLANE AND LINEAR GRAPHS:

An Ordered Pair of Real Numbers:

An ordered pair of real numbers x and y is a pair (x, y) in which elements are written in specific order. e.g., (x, y) is an ordered pair in which first element is x and second is y and $(x, y) \neq (y, x)$. e.g., (2, 3) and (3, 2) are two different ordered pairs.

$(x, y) = (m, n)$ only if $x = m$ and $y = n$.

Recognizing an Ordered Pair:

In the class room the seat of a student is the example of an ordered pair. For example, the seat of the student A is 5th place in the 3rd row, so it corresponds to the ordered pair (3, 5). Here '3' show the number of the row and '5' shows its seat number in this row.

Similarly an ordered pair (4, 3) represents a seat located to a student A in the examination hall is at the 4th row and 3rd column i.e. 3rd place in the 4th row.

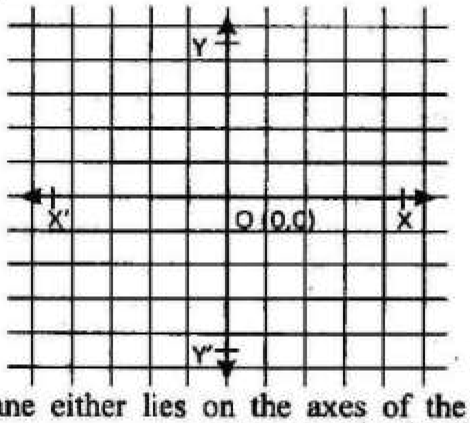
Cartesian Plane:

The Cartesian plane establishes one-to-one correspondence between the set of ordered pairs $R \times R = \{(x, y) \setminus x, y \in R\}$ and the points of the Cartesian plane.

In plane two perpendicular straight lines are drawn. The lines are called the coordinate axes. The point O, where the two lines meet is called origin. This plane is called the coordinate plane or the Cartesian plane.

Identification of Origin and Co-ordinate Axes:

The horizontal line XOX' is called the x-axis and the vertical line YOY' is called the y-axis. The point O where the x-axis and y-axis meet is called the origin and it is denoted by O (0, 0).

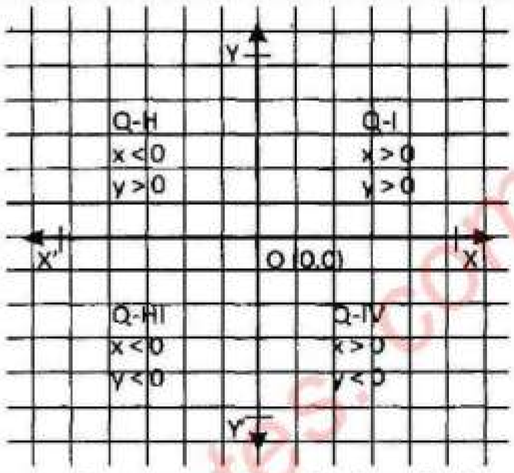


We have noted that each point in the plane either lies on the axes of the coordinate plane or in any one of quadrants of the plane namely XOY, YOX′, X′OY′ and Y′OX called the first, second, third and the fourth quadrants of the plane subdivided by the coordinate axes of the plane. They are denoted by Q-I, Q-II, Q-III and Q-IV respectively.

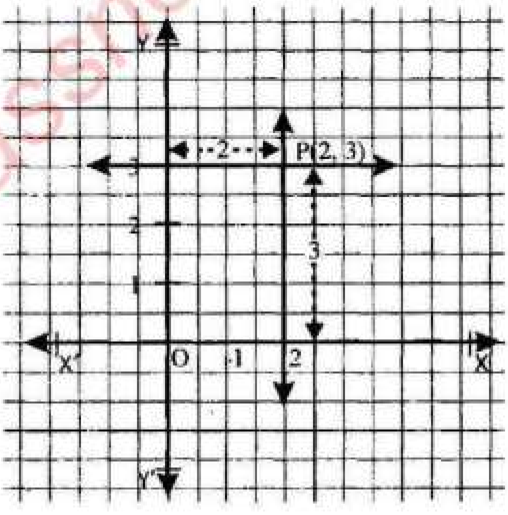
The signs of the coordinates of the points (x, y) are shown below;

For Example:

(1) The point $(-3, -1)$ lies in Q-III.
(2) The point $(2, -3)$ lies in Q-IV.
(3) The point $(2, 5)$ lies in Q-I.
(4) The point $(2, 0)$ lies on x-axis.



Location of the Point P(a , b) in the Plane Corresponding to the Ordered Pair (a, b):



Let (a, b) be an ordered pair of R×R.

In the reference system, the real number 'a' is measured along x-axis, 'a' units away from the origin along OX (if a > 0) and the real number 'b' along y-axis, 'b' units away from the origin along OY (if b > 0). From 'b' on OY, draw the line parallel to x-axis and from 'a' on OX draw line parallel to y-axis. Both the lines meet at the point P. Then the point P corresponds to the ordered pair (a, b).

In the graph shown above 2 is the X-coordinate and 3 is the y-coordinate of the point P which is denoted by P (2, 3).

In this way coordinates of each point in the plane are obtained.

The X-coordinate of the point is called **abscissa** of the point P(x, y) and the Y-coordinate is called its **ordinate.**

1. Each point P of the plane can be identified by the coordinates of the pair (x, y) and is represented by P(x, y).
2. All the points of the plane have y-coordinate, $y = 0$ if they lie on the x-axis. i.e.,
   P $(-2, 0)$ lies on the axis.
3. All the points of the plane have x-coordinate $x = 0$ if they lie on the y-axis, i.e., Q (0, 3) lies on the y-axis.

Drawing different Geometrical Shapes of Cartesian Plane:

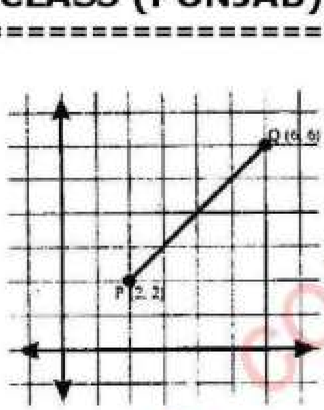
We define first the idea of collinear points before going to form geometrical shapes.

(a) Line-Segment:

Example-1:

Let P (2,2) and Q(6, 6) are two points.

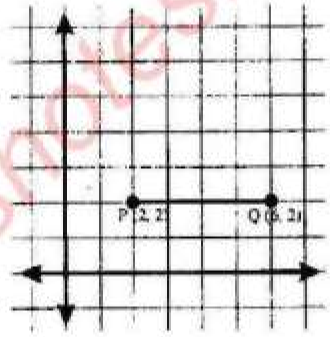
1. Plot points P and Q.
2. Join the points P and Q, we get the line segment PQ. It is represented by PQ.



Example 2:

Plot points P(2, 2) and Q(6, 2). By joining them, we get a line segment PQ parallel to x-axis.

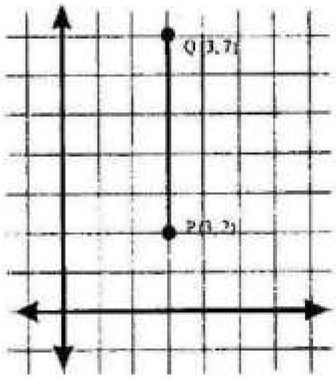
Where ordinate of both points is equal.



Example-3:

Plot points P(3, 2) and Q(3, 7). By joining them, we get a line segment PQ parallel to y-axis.

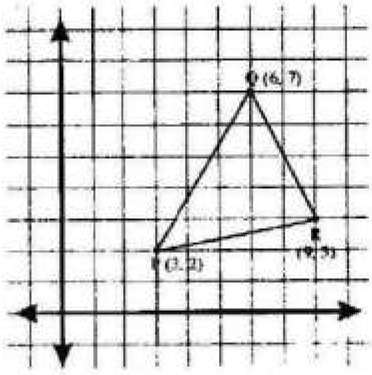
In this graph abscissa of both the points are equal.



(b) Triangle

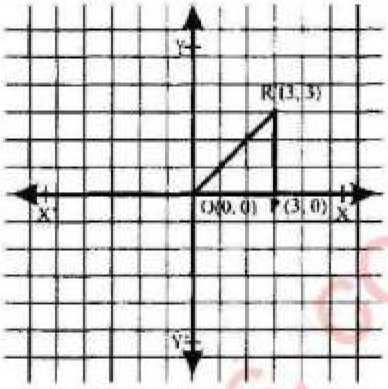
Example-1:

Plot the points P(3, 2), Q(6, 7) and R.(9, 3). By joining them, we get a triangle PQR.



Example 2:

For points O(0, 0), P(3, 0) and R(3, 3). The triangle OPR is constructed as shown by the side.

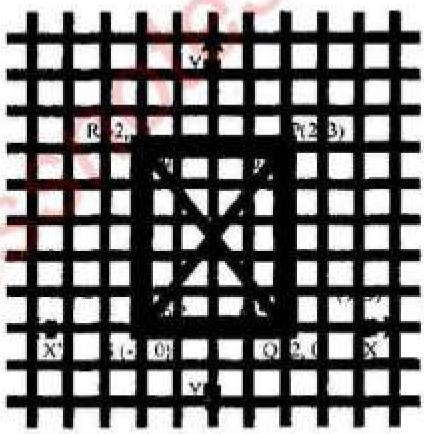


(c) Rectangle

Example:

Plot the points P(2, 3), Q(2, 0), R(-2, 3) and S (-2, 0). Joining the points P, Q, R and S, we get a rectangle PQRS.

Along y-axis, 2(length of square) =1 unit



Construction of a Table for Pairs of Values Satisfying a Linear Equation in Two Variables:

Let $2x + y = 1$ ______(i)

be a linear equation in two variables x and y.

The ordered pair (x, y) satisfies the equation and by varying x, corresponding y is obtained.

We express eq. (i) in the form, we get

$y = -2x + 1$ _____(ii)

The pairs (x, y) which satisfy (ii) are tabulated below.

x	y	(x, y)
−1	3	(−1,3)
0	1	(0, 1)
1	−1	(1,−1)
3	−5	(3,−5)

At $x = -1$ $y = (-2)(-1) + 1 = 2 + 1 = 3$

At $x = 0$ $y = (-2)(0) + 1 = 0 + 1 = 1$

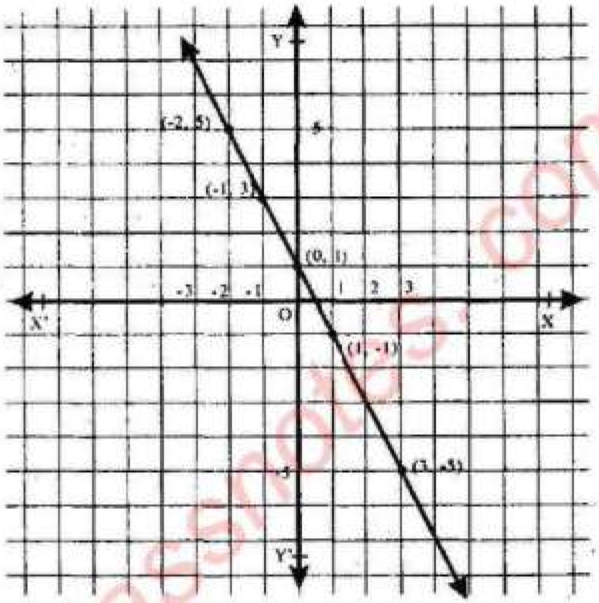
At $x = 1$ $y = (-2)(1) + 1 = -2 + 1 = -1$

At $x = 3$ $y = -6 + 1 = -5$

Similarly all the points can be computed, the ordered pairs of which do satisfy the equation (i).

Plotting the points to get the graph:

Now we plot the points obtained in the table. Joining these points we get the graph of the equation. The graph of $y = -2x + 1$ is shown in the following figure.



Scale of Graph:

To draw the graph of an equation we choose a scale e.g., 1 cm represents 5 meters or 1 small square represents 10 or 5 meters. It is selected by keeping in mind the size of the paper. Sometimes the same scale is used for both x and y coordinates and some times we use different scales for x and y-coordinate depending on the values of the coordinates.

Drawing Graphs of the following Equations:

(a) $y = c$, where c is constant.
(b) $x = a$, where a is constant.
(c) $y = mx$, where m is constant.
(d) $y = mx + c$, where m and c both are constants.

By drawing the graph of an equation is meant to plot those points in the plane, which form the graph of the equation (by joining the plotted points).

(a) The equation $y = c$ is formed in the plane by the set.

$S = \{(x, c): x \text{ lies on the x-axis}\} \subseteq R \times R$.

The procedure is explained with the help of following examples.

Consider the equation $y = 2$

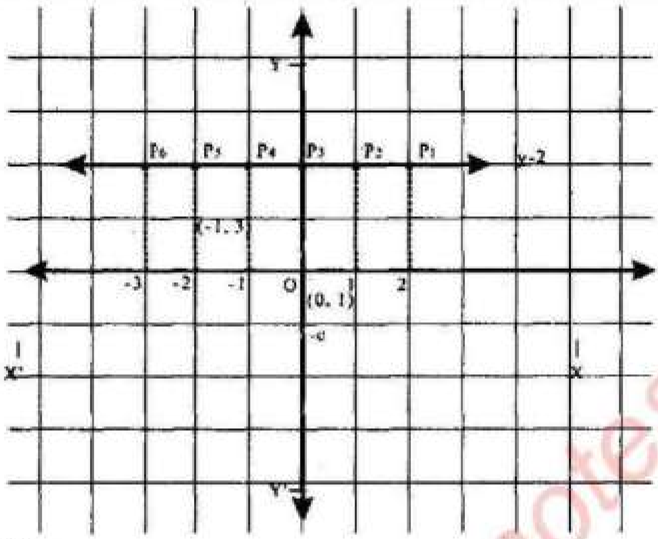
The set S is tabulated as;

x	....	−3	−2	−1	0	1	2	....
y	2	2	2	2	2	2	2	2

The points of S are plotted in the plane,

Similarly graph of $y = -4$ is shown as:

So, the graph of the equation of the type y = is obtained as:



(i) the straight line

(ii) the line is parallel to x-axis

(iii) the line is above the x-axis at a distance c units if $c > 0$.

(iv) the line (shown as $y = -4$) is below the x-axis at the distance c units as $c < 0$.

(v) the line is that of x-axis at the distance c units if $c = 0$,

(b) The equation, $x = a$ is drawn in the plane by the points of the set

$S = \{(a, y) : y \in R\}$

The points of S are tabulated as follows:

X	a	a	a	a	a	a	a	a	...
Y	...	−2	−1	0	1	2	3	4	...

The points of S are plotted in the plane as,..... (a,−2), (a, −1), (a, 0), (a, 1), (a, 2),.... etc.

The point (a, 0) on the graph of the equation $x = a$, lies on the x-axis while (a, y) is above the x-axis if $y > 0$ and below the x-axis if $y < 0$. By joining the points, we get the line.

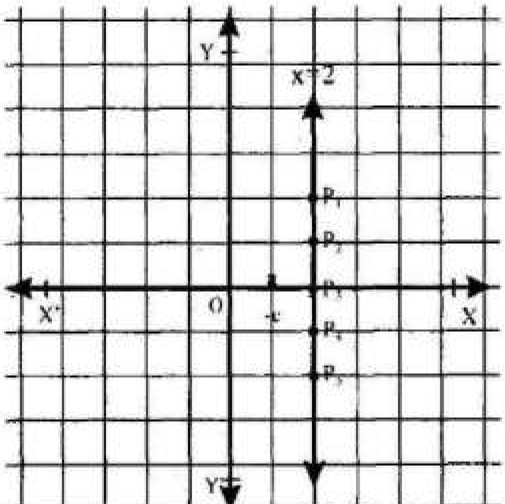
The procedure is explained with the help of following examples.

Consider the equation $x = 2$.

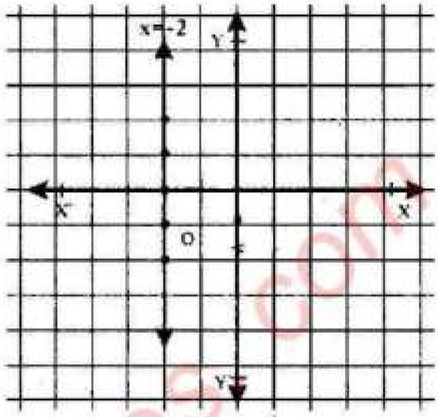
Table for the points of equation is as under

x	2	2	2	2	2	2	...2...
y	...	−2	−1	0	1	2	...

Thus, graph of the equation $x = 2$ is shown as:



Similarly graph for equation $x = -2$ is shown as



So, the graph of the equation of the type $y = a$ is obtained as:

(i) straight line
(ii) the line parallel to the y-axis.
(iii) the line is on the right side of y-axis at distance "a" units if $a > 0$.
(iv) the line $x = -2$ is on the left side of y-axis at the distance a units as $a < 0$.
(v) the line is y-axis if $a = 0$.

(c) The equation $y = mx$, (for a fixed $m \in R$) is formed by the points of the set

$W = \{(x, mx): x \in R\}$

i.e., $W = \{..., (-2, -2m), (-1, -m), (0, 0), (1, m), (2,2m),....\}$.

The points corresponding to the ordered pairs of the set W are tabulated below:

x	......	-2	-1	0	1	2	......
y	......	$-2m$	$-m$	0	m	2m	......

The procedure is explained with the help of following examples.

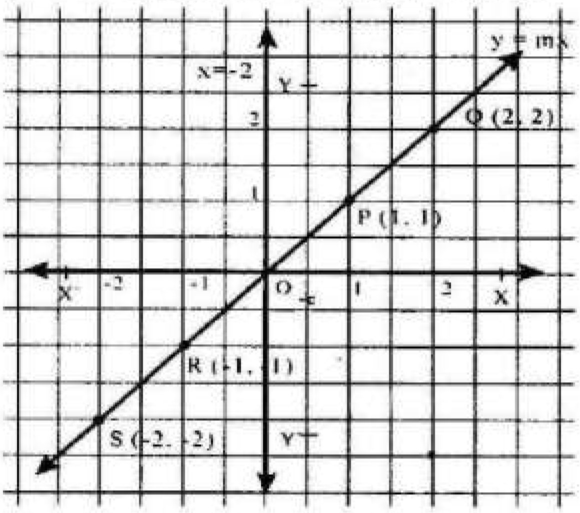
Consider the equation $y = x$, where $m = 1$

Table of points for equation is as under:

x	...	-2	-1	0	1	2	...
y	...	-2	-1	0	1	2	...

By joining the plotted points the graph of the equation of the type $y = mx$ is,

(i) the straight line
(ii) it passes through the origin O (0, 0)
(iii) m is the slope of the line
(iv) the graph of line splits the plane into two equal parts. If $m = 1$ then the line becomes the graph of the equation $y = x$.
(v) If $m = -1$ then line is the graph of the equation $y = -x$.
(vi) the line meets both the axes at the origin and no other point.



(d) Now we move to a generalized form of the equation, i.e.,

$y = mx + c$, where $m, c \neq 0$.

The points corresponding to the ordered pairs of the set

$S = \{(x, mx + c): m, c (\neq 0) \in R\}$ are tabulated below

x	0	1	2	3	......	x	......
y	c	m + c	2m + c	3m + c	......	mx + c	......

The procedure is explained with the help of following examples.

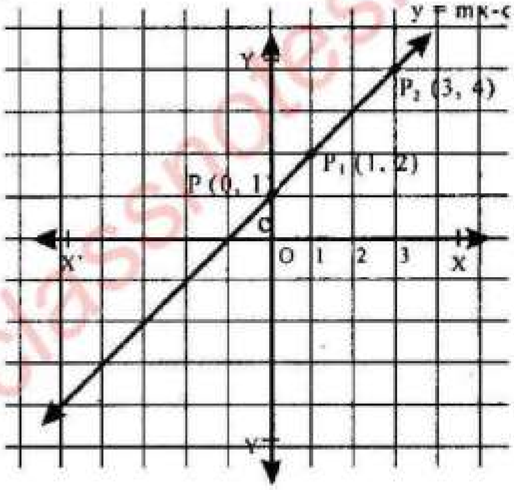
Consider the equation

$y = x + 1$, where $m = 1$, $c = 1$

We get the table

x	...0	1	2	3	...	...	...
y	... 1	2	3	4	...	...	.....

These points are plotted in plane as below:



We see that

(i) $y = mx + c$ represents the graph of a line.

(ii) It does not pass through the origin O (0, 0).

(iii) It has intercept c units along the y-axis away from the origin.

(iv) m is the slope of the line whose equation is $y = mx + c$.

In particular if,

(i) $c = 0$ then $y = mx$ passes through the origin.

(ii) If $m = 0$, then the line $y = c$ is parallel to x-axis.

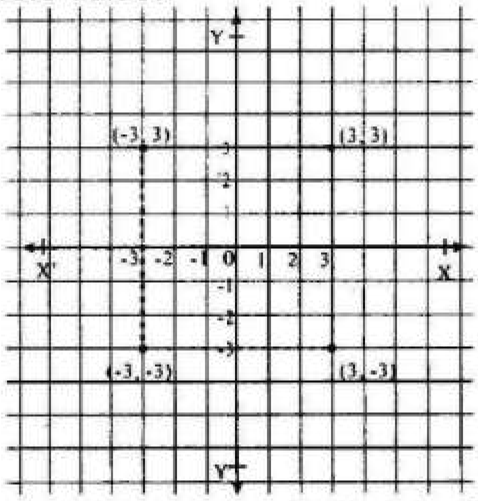
Drawing Graph from a given Table of Discrete Values:

If the points are discrete the graph is just the set of points. The points are not joined.

For example, the following table of discrete values is plotted as:

x	3	3	−3	−3
y	3	−3	3	−3

So, the dotted square shows the graph of discrete values.



Solving Real Life Problems:

We often use the graph to solve the real life problems. With the help of graph, we can determine the relation or trend between the both quantities.

We learn the procedure of drawing graph of real life problems with the help of following examples.

Example: Equation y = x + 16 shows the relationship between the age of father and son i.e. if the age of son is x, then the father's age is y. Draw the graph.

Solution:

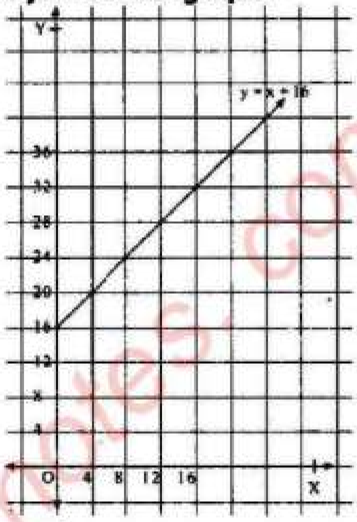
We know that

$y = x + 16$

Table of points for equation is given as:

x	0	4	8	12	16	......
x	16	20	24	28	32	......

By plotting the points we get the graph of a straight line as shown in the figure.



Solved Exercise 8.1

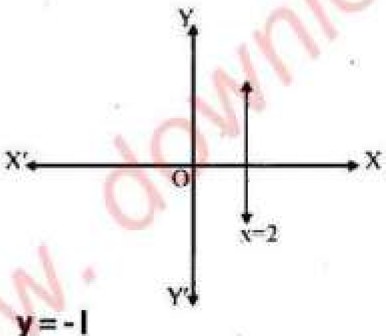
1. Determine the quadrant of the coordinate plane in which the following points lie: P(-4, 3), Q(-5, -2), R(2, 2) and S(2, -6).

Solution: P (–4, 3) is in II quadrant. Q (-5, -2) is in III quadrant.
R (2, 2) is in I quadrant. S (2, -6) is in IV quadrant.

2. Draw the graph of each of the following.

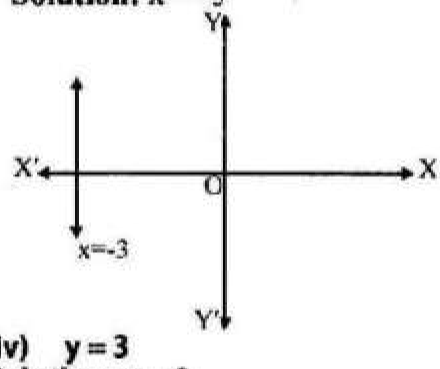
(i) x = 2

Solution: x = 2



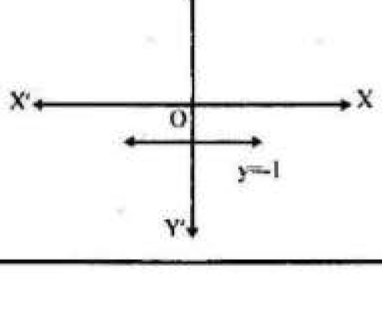
(ii) x = - 3

Solution: x = -3



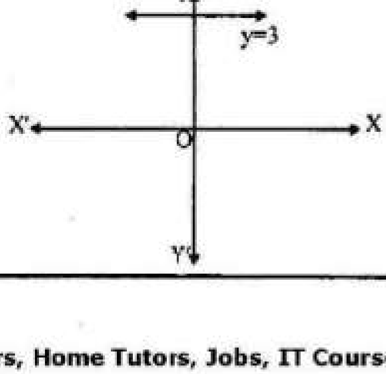
(iii) y = - 1

Solution: y = -1



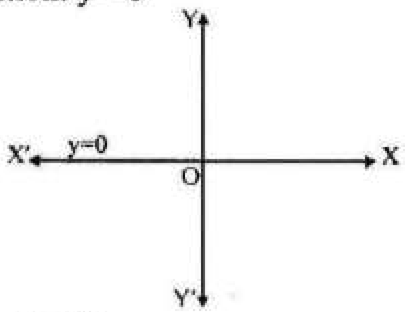
(iv) y = 3

Solution: y = 3



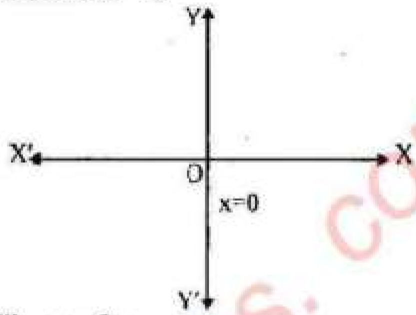
(v) $y = 0$

Solution: $y = 0$



(vi) $x = 0$

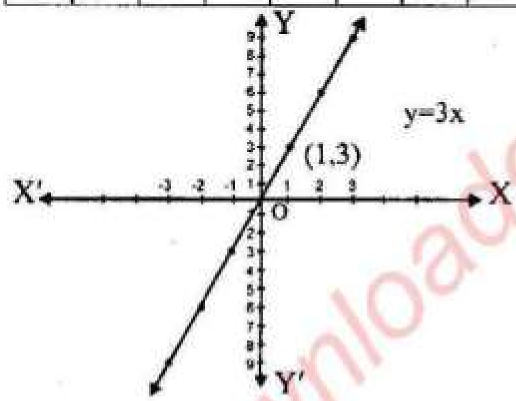
Solution: $x = 0$



(vii) $y = 3x$

Solution: $y = 3x$

x	-3	-2	-1	0	1	2	3
y	-9	-6	-3	0	3	6	9

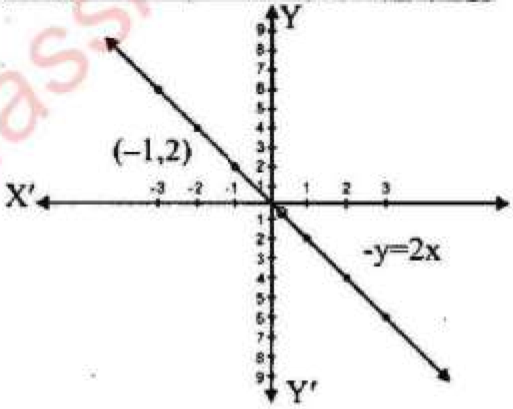


(viii) $-y = 2x$

Solution: $-y = 2x$

$y = -2x$

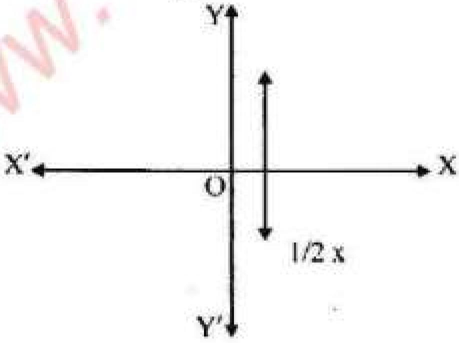
x	-3	-2	-1	0	1	2	3
y	+6	+4	+2	0	-2	-4	-6



(ix) $\frac{1}{2} = x$

Solution: $\frac{1}{2} = x$

$x = \frac{1}{2}$

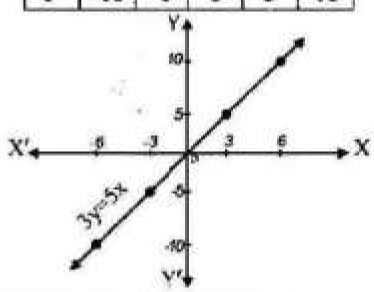


(x) $3y = 5x$

Solution: $3y = 5x$

$y = \frac{5x}{3}$

x	-6	-3	0	3	6
y	-10	-5	0	5	10

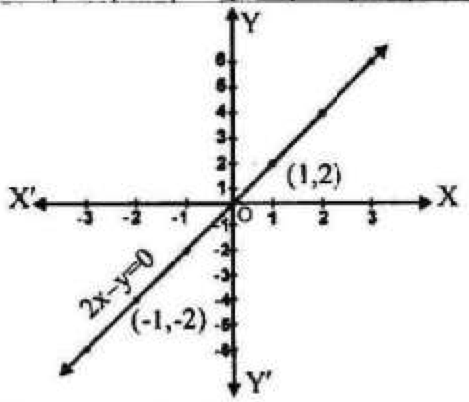


(xi) $2x - y = 0$

Solution: $2x - y = 0$

$y = 2x$

x	-3	-2	-1	0	1	2	3
y	-6	-4	-2	0	2	4	6

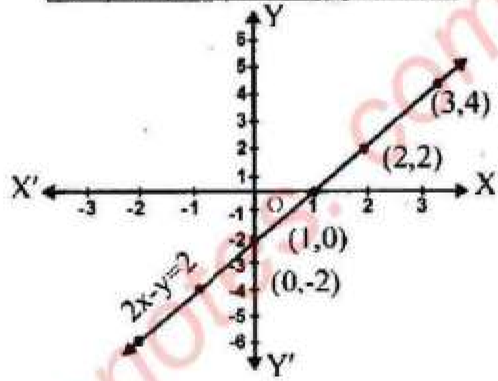


(xii) $2x - y = 2$

Solution: $2x - y = 2$

$y = 2x - 2$

x	-2	-1	0	1	2	3
y	-6	-4	-2	0	2	4



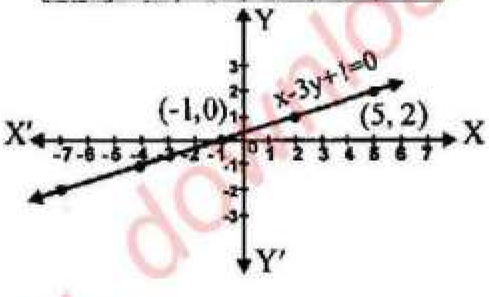
(xiii) $x - 3y + 1 = 0$

Solution: $x - 3y + 1 = 0$

$3y = x + 1$

$y = \frac{1}{3}(x + 1)$

x	-7	-4	-1	2	5
y	-2	-1	0	1	2



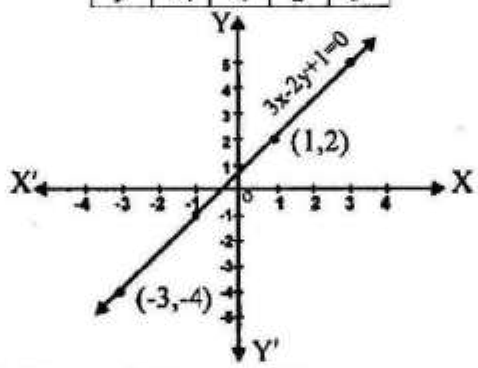
(xiv) $3x - 2y + 1 = 0$

Solution: $3x - 2y + 1 = 0$

$2y = 3x + 1$

$y = \frac{1}{2}(3x + 1)$

x	-3	-1	1	3
y	-4	-1	2	5



3. Are the following lines (i) parallel to x-axis (ii) parallel to y-axis?

(i) $2x - 1 = 3$ (ii) $x + 2 = -1$ (iii) $2y + 3 = 2$

(iv) $x + y = 0$ (v) $2x - 2y = 0$

(i) Solution:

$2x - 1 = 3 \Rightarrow 2x = 3 + 1 \Rightarrow 2x = 4 \Rightarrow x = 2$

The above line is parallel to y-axis.

(ii) Solution:

$x + 2 = -1 \Rightarrow x = -2 - 1 \Rightarrow x = -3$

The above line is parallel to y-axis.

(iii) Solution:

$2y + 3 = 2 \Rightarrow 2y = -3 + 2 \Rightarrow 2y = -1 \Rightarrow y = -\frac{1}{2}$

The above line is parallel to x-axis.

(iv) Solution:

$x + y = 0 \Rightarrow x = -y$

The above line is neither parallel to x-axis nor parallel to y-axis.

(v) Solution:

$2x - 2y = 0 \Rightarrow 2x = 2y \Rightarrow x = y$

The above line is neither parallel to x-axis nor parallel to y-axis.

4. Find the value of m and c of the following lines by expressing them in the form $y = mx + c$.

(a) $2x + 3y - 1 = 0$ (b) $x - 2y = -2$ (c) $3x + y - 1 = 0$

(d) $2x - y = 7$ (e) $3 - 2x + y = 0$ (f) $2x = y + 3$

(a) Solution:

$2x + 3y - 1 = 0 \Rightarrow 3y = -2x + 1 \Rightarrow y = -\frac{2}{3}x + \frac{1}{3}$

Compare it with

$y = mx + c \Rightarrow m = -\frac{2}{3}$ and $c = \frac{1}{3}$

(b) Solution:

$x - 2y = -2 \Rightarrow -2y = -x - 2 \Rightarrow y = \frac{1}{2}x + 1$

Compare it with

$y = mx + c \Rightarrow m = \frac{1}{2}$ and $c = 1$

(c) Solution:

$3x + y - 1 = 0 \Rightarrow y = -3x + 1$

Compare it with

$y = mx + c \Rightarrow m = -3$ and $c = 1$

(d) Solution:

$2x - y = 7 \Rightarrow -y = -2x + 7 \Rightarrow y = 2x - 7$

Compare it with

$y = mx + c \Rightarrow m = 2$ and $c = -7$

(e) Solution:

$3 - 2x + y = 0 \Rightarrow y = 2x - 3$

Compare it with

$y = mx + c \Rightarrow m = 2$ and $c = -3$

(f) Solution:

$2x = y + 3 \Rightarrow y = 2x - 3$

Compare it with

$y = mx + c \Rightarrow m = 2$ and $c = -3$

5. Verify whether the following point lies on the line $2x - y + 1 = 0$ or not.

(i) (2, 3) (ii) (0, 0) (iii) (-1, 1) (iv) (2, 5) (v) (5, 3)

(i) Solution:

$2x - y + 1 = 0$

Put (2, 3) in the above equation, we get
$2(2)-(3)+1=0 \Rightarrow \quad 4-3+1=0 \quad \Rightarrow \quad 5-3=0$
$\Rightarrow \quad 2 \neq 0$ (Which is not true)
Hence, (2, 3) does not lie on the given line.

(ii) Solution:
$2x-y+1=0$
Put (0, 0) in the above equation, we get
$2(0)-(0)+1=0 \Rightarrow \quad 0-0+1=0 \quad \Rightarrow \quad 0+1=0$
$\Rightarrow \quad 1 \neq 0$ (Which is not true)
Hence, (0, 0) does not lie on the given line.

(iii) Solution:
$2x-y+1=0$
Put (−1, 1) in the above equation, we get
$2(-1)-(1)+1=0 \quad \Rightarrow \quad -2-1+1=0 \quad \Rightarrow \quad -3+1=0$
$\Rightarrow \quad -2 \neq 0$ (Which is not true)
Hence, (−1, 1) does not lie on the given line.

(iv) Solution:
$2x-y+1=0$
Put (2, 5) in the above equation, we get
$2(2)-(5)+1=0 \Rightarrow \quad 4-5+1=0 \quad \Rightarrow \quad 5-5=0$
$\Rightarrow \quad 0=0$ (Which is true)
Hence, (2, 5) lies on the given line.

(v) Solution:
$2x-y+1=0$
Put (5, 3) in the above equation, we get
$2(5)-(3)+1=0 \Rightarrow \quad 10-3+1=0 \quad \Rightarrow \quad 11-3=0$
$\Rightarrow \quad 8 \neq 0$ (Which is not true)
Hence, (5, 3) does not lie on the given line.

CONVERSION GRAPHS

To Interpret Conversion Graph:

In this section we shall consider conversion graph as a linear graph relating to two quantities which are in direct proportion.

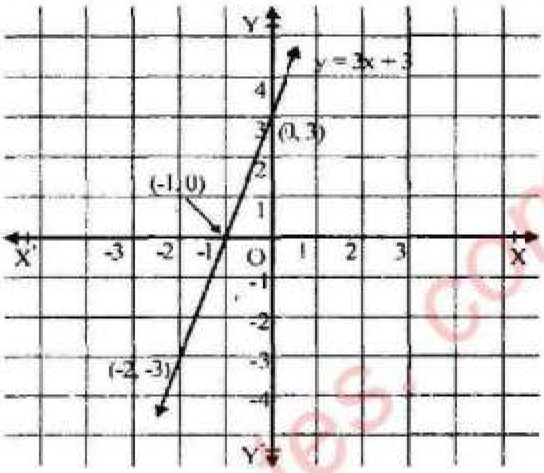
Let $y=f(x)$ be an equation in two variables x and y.

We demonstrate the ordered pairs which lie on the graph of the equation $y = 3x+3$ and are tabulated below:

x	...0	−1	−2...
y	...3	0	−3 ...
(x, y)	... (0, 3)	(−1,0)	(−2, −3)...

By plotting the points in the plane corresponding to the ordered pairs (0, 3), (−1,0)

and (–2, –3) etc. we form the graph of the equation $y = 3x + 3$.



Reading a Given Graph:

From the graph of $y = 3x + 3$ as shown above.

(i) for a given value of x we can read the corresponding value of y with the help of equation $y = 3x + 3$, and

(ii) for a given value of y we can read the corresponding value of x, by converting equation $y = 3x + 3$ to equation $x = \frac{1}{3}y - 1$ and draw the corresponding conversion graph.

In the conversion graph we express x in terms of y as explained below.

$y = 3x + 3$

$\Rightarrow \quad y - 3 = 3x + 3 - 3$

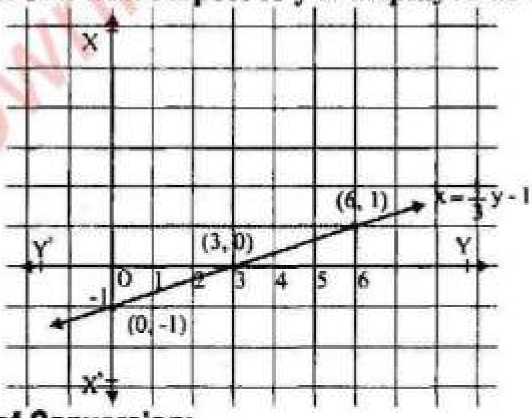
$\Rightarrow \quad y - 3 = 3x \quad \text{or } 3x = y - 3$

$\Rightarrow \quad x = \frac{1}{3}y - 1$, where x is expressed in terms of y.

We tabulate the values of the dependent variable x at the values of y.

y	3	0	6....
x	 0	–1	1 ...
(y, x)	.... (3,0)	(0,–1)	(6, 1) ...

The conversion graph of x with respect to y is displayed as below:



Reading the Graphs of Conversion:

(a) Example: (Kilometer (Km) and Mile (M) Graphs):

To draw the graph between kilometer (Km) and Miles (M), we use the following relation:

One kilometer = 0.62 miles (approximately)

And one mile = 1.6 km (approximately)

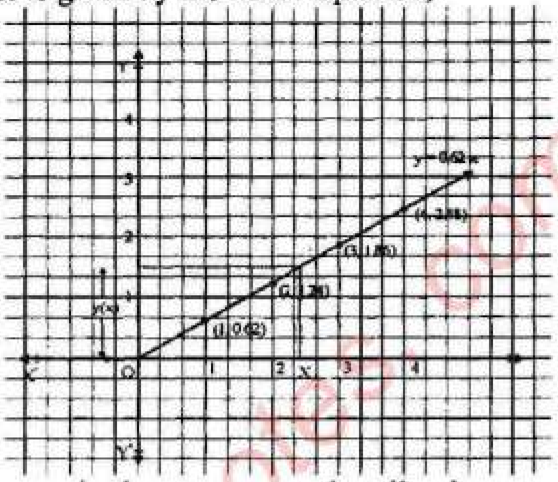
(i) The relation of mile against kilometer is given by the linear equation,

$y = 0.62x$

If y is a mile and x, a kilometer, then we tabulate the ordered pairs (x, y) as below;

x	0	1	2	3	4...
y	0	0.62	1.24	1.86	2.48...

The ordered pairs (x,y) corresponding to $y = 0.62x$ are represented in the Cartesian plane. By joining them we get' the desired following graph of miles against kilometers.



For each quantity of kilometer x along x-axis there corresponds mile along y – axis.

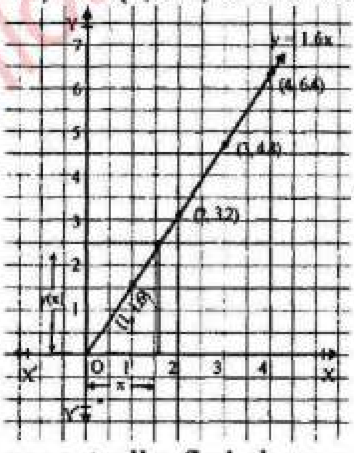
(ii) The conversion graph of kilometer against mile is given by

$y = 1.6x$ (approximately)

If y represents kilometers and x a mile, then the values x and y are tabulated as:

x	0	1	2	3	4 ...
y	0	1.6	3.2	4.8	6.4...

We plot the points in the xy-plane corresponding to the ordered pairs.
(0, 0), (1, 1.6), (2, 3.2), (3, 4.8) and (4, 6.4) as shown in figure.



By joining the points we actually find the conversion graph of kilometres against miles.

(b) Conversion Graph of Hectares and Acres:

(i) The relation between Hectare and Acre is defined as:

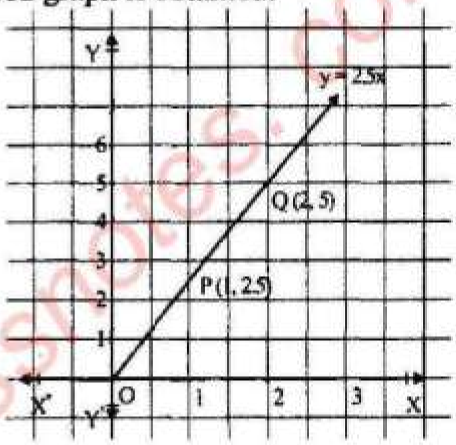
$$\text{Hectare} = \frac{640}{259} \text{ Acres} = 2.5 \text{ Acres (approximately)}$$

In case when hectare = y and acre = x, then $y = 2.5x$

If x is represented as hectare along the horizontal axis and y as Acre along y-axis, the values are tabulated below:

x	0	1	2	3	4...
y	0	2.5	5.0	7.5	10...

The ordered pairs (0, 0), (1, 2.5), (2, 5) etc., are plotted as points in the xy-plane as below and by joining the points the required graph is obtained:



(ii) Now the conversion graph Acre = $\frac{1}{2.5}$ Hectare is simplified as,

Acre = $\frac{10}{25}$ Hectare = 0.4 Hectare (approximately)

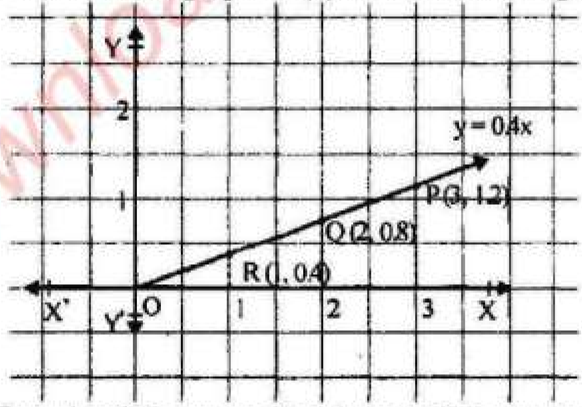
If Acre is measured along x-axis and hectare along y-axis then

$y = 0.4x$

The ordered pairs are tabulated in the following table:

x	0	1	2	3 ....
y	0	0.4	0.8	1.2...

The corresponding ordered pairs (0, 0), (1, 0.4), (2, 0.8) etc.., are plotted in the xy-plane, join of which will form the graph of (b) as a conversion graph of (a):



(c) Conversion Graph of Degrees Celsius and Degrees Fahrenheit:

(i) The relation between degree Celsius (C) and degree Fahrenheit (F) is given by

$$F = \frac{9}{5}C + 32$$

The values of F at C = 0 is obtained as

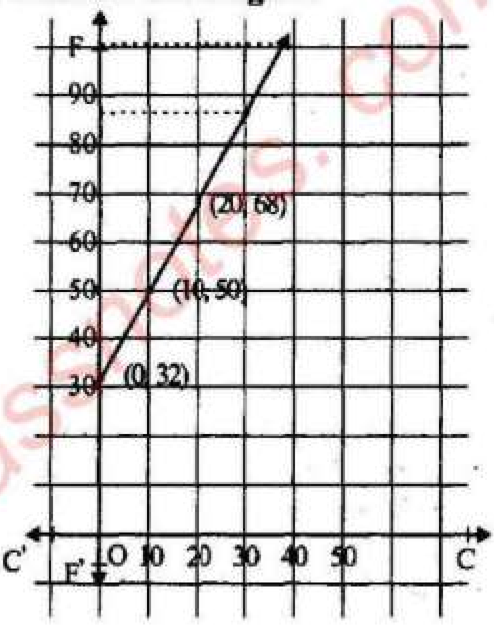
$$F = \frac{9}{5} \times 0 + 32 = 0 + 32 = 32 \qquad \text{Similarly } F = \frac{9}{5} \times 10 + 32 = 18 + 32 = 50$$

$F = \frac{9}{5} \times 20 + 32 = 36 + 32 = 68$ $\quad$ $F = \frac{9}{5} \times 100 + 32 = 180 + 32 = 212$

We tabulate the values of C and F.

C	0°	10°	20°	50°	'100°...
F	32°	50°	68°	122°	212°...

The conversion graph of F with respect to C is shown in the figure.



Note from the graph that the value of C corresponding to

(i) $F = 86°$ is $C = 30°$ and

(ii) $F = 104°$ is $C = 40°$.

(iii) Now we express C in terms of F for the conversion graph of C with respect to F as below:

$$C = \frac{5}{9}\ (F - 32)$$

The values for F = 68° and F =176° are

$C = \frac{5}{9}\ (68 - 32) = \frac{5}{9} \times 36 = 20°$ and

$C = \frac{5}{9}(176 - 32) = \frac{5}{9}(144) = 5\times16 = 80°$

Find out at what temperature will the two readings be same?

i.e., $F = \frac{5}{9}C + 32$

$\Rightarrow \left(\frac{5}{9} - 1\right) C = -32 \Rightarrow \frac{4}{5} C = -32 \Rightarrow C = \frac{-32\times5}{4} = -40$

To verify at $C = -40$, we have

$F = \frac{9}{5} \times (-40) + 32 = 9(-8) + 32 = -72 + 32 = -40°$

(d) Conversion Graph of US$ and Pakistani Currency:

The Daily News, on a particular day informed the conversion rate of Pakistani currency to the US$ currency as,

<u>1 US$ = 66.46 Rupees</u>

If the Pakistani currency y is an expression of US$ x, expressed under the rule $y = 66.46\,x = 66x$ (approximately)

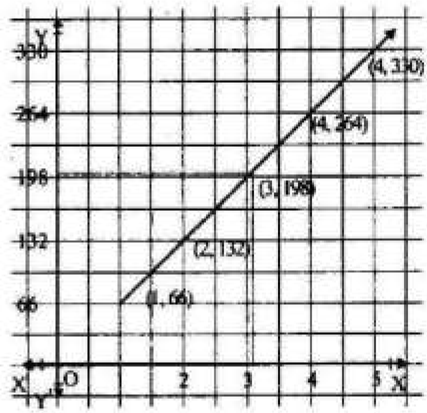
Then draw the conversion graph.

We tabulate the values as below.

x	1	2	3.	4...
y	66	132	198	264...

Plotting the points corresponding to the ordered pairs (x, y) from the above

table and joining them provides the currency linear graph of rupees against dollars as shown in the figure.



Conversion graph $x = \frac{1}{66}$ y of y = 66x can be shown by interchanging x-axis to y-axis and vice versa.

Solved Exercise 8.2

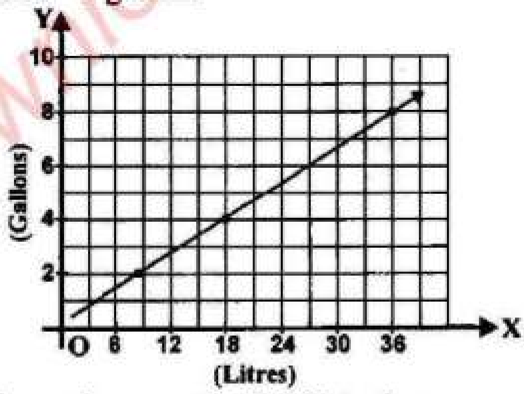
1. Draw the conversion graph between litres and gallons using the relation 9 litres = 2 gallons (approximately), and taking litres along horizontal axis and gallons along vertical axis. From the graph, read

(i) the number of gallons in 18 litres (ii) the number of litres in 8 gallons

Solution: As 9 litres = 2 gallons

(i) 18 litres = 4 gallons

(ii) 36 litres = 8 gallons



2. On 15.03.2008 the exchange rate of Pakistani currency and Saudi Riyal was as under:

1 S. Riyal = 16.70 Rupees

If Pakistani currency y is an expression of S. Riyal x, expressed under the rule y=16.70x, then draw the conversion graph between these two currencies by taking S. Riyal along x-axis.

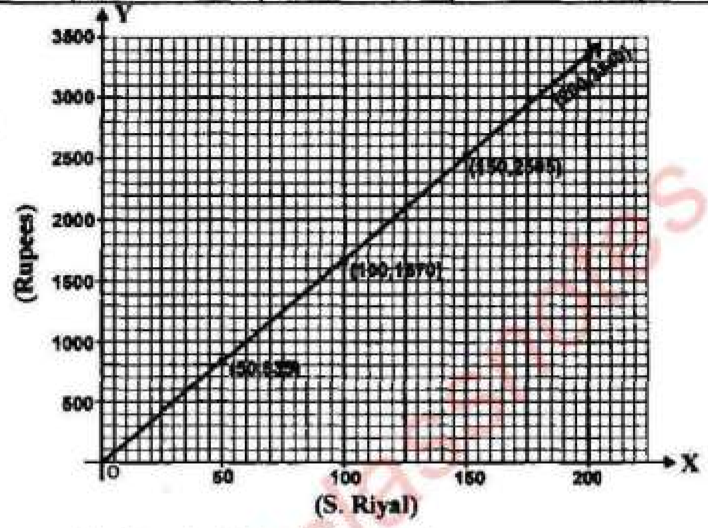
Solution: As 1 S. riyal = 16.70 rupees

$y = 16.70\,x$

Pakistani Currency = y, S. Riyal = x

We tabulate the values as below.

X	50	100	150	200
Y	835	1670	2505	3340



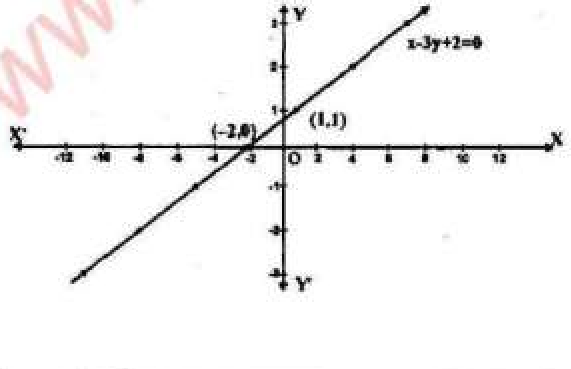
3. Sketch the graph of each of the following lines.

(a) $x - 3y + 2 = 0$

Solution: $x - 3y + 2 = 0$

$x = 3y - 2$

Y	-3	-2	-1	0	1	2	3
X	-11	-8	-5	-2	1	4	7



(b) $3x - 2y - 1 = 0$

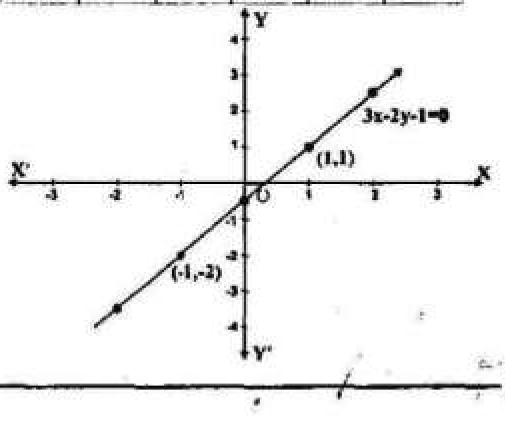
Solution: $3x - 2y - 1 = 0$

$-2y = -3x + 1$

$y = \frac{-3}{-2}x + \frac{1}{-2}$

$y = \frac{3}{2}x - \frac{1}{2}$

x	-2	-1	0	1	2
y	-3.5	-2	-0.5	1	2.5



(c) $2y - x + 2 = 0$

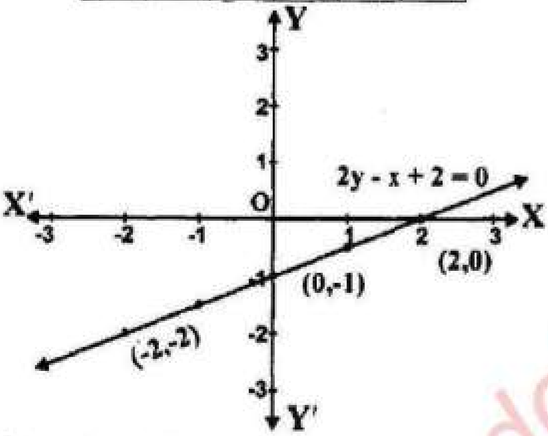
Solution: $2y - x + 2 = 0$

$2y = x - 2$

$y = \frac{1}{2}x - \frac{2}{2}$

$y = \frac{1}{2}x - 1$

x	-2	-1	0	1	2
y	-2	-1.5	-1	-0.5	0

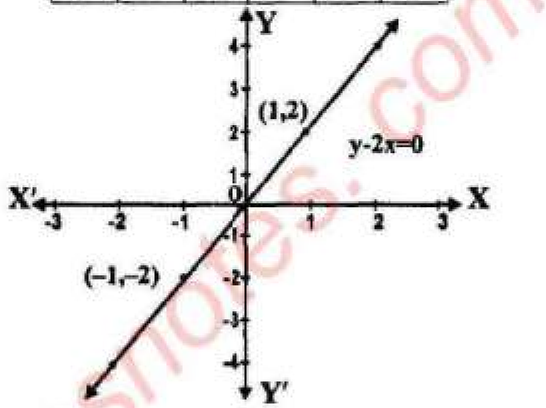


(d) $y - 2x = 0$

Solution: $y - 2x = 0$

$y = 2x$

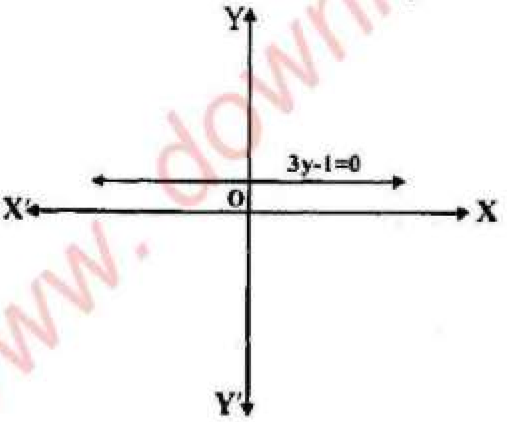
x	-2	-1	0	1	2
y	-4	-2	0	2	4



(e) $3y - 1 = 0$

Solution: $3y - 1 = 0$

$3y = 1 \quad \Rightarrow y = \frac{1}{3}$

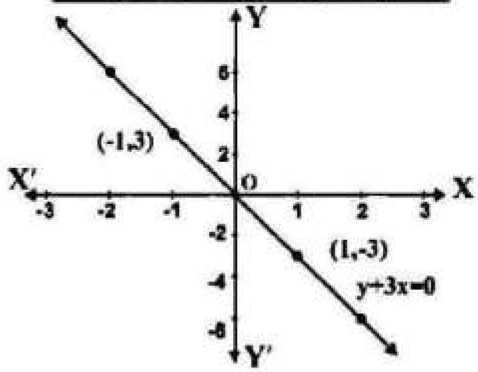


(f) $y + 3x = 0$

Solution: $y + 3x = 0$

$y = -3x$

x	-2	-1	0	1	2
y	6	3	0	-3	-6

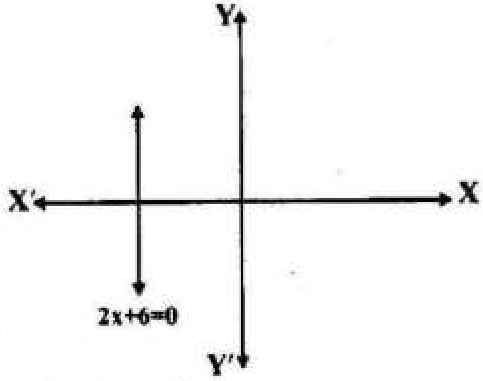


(g) $2x + 6 = 0$

Solution: $2x + 6 = 0$

$2x = -6$

$x = -3$



4. Draw the graph for following relations.

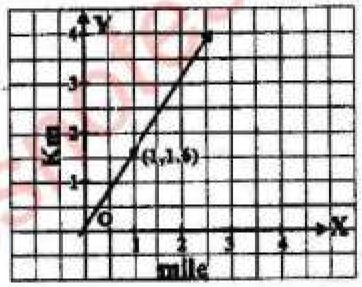
(i) One mile = 1.6 km

Solution: One mile = 1.6 km

Let, miles = x and kilometer = y

So, x = 1.6y

x	0	1.6	3.2
y	0	1	2



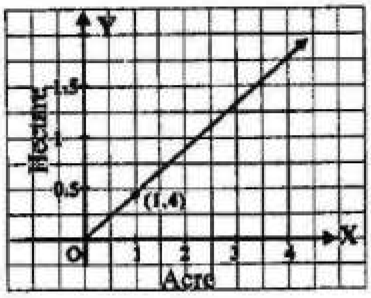
(ii) One Acre =0.4 Hectare

Solution: One Acre =0.4 Hectare

Let, Acre = x and Hectare = y

So, x = 0.4y

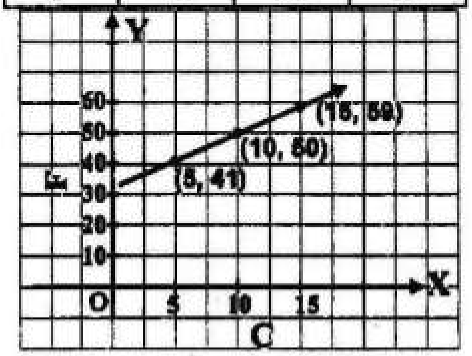
x	0	0.4	0.8	1.2
y	0	1	2	3



(iii) $F = \frac{9}{5}C + 32$

Solution: $F = \frac{9}{5}C + 32$

C	5	10	15
F	41	50	59



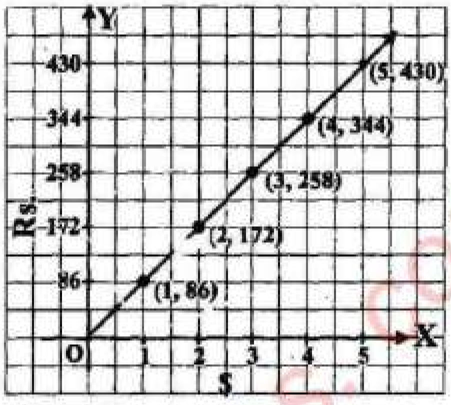
(iv) One Rupee = $\frac{1}{86}$ \$

Solution: Given Rs.1 = $\frac{1}{86}$ \$

So Rs.86 = 1 \$

We tabulate the values as below.

x (\$)	1	2	3	4	5
y (Rs.)	86	172	258	344	430



Graphical Solution of Linear Equations in two Variables:

We solve here simultaneous linear equations in two variables by graphical method.

Let the system of equations be,

$2x - y = 3$ …………… (i) $x + 3y = 3$ …………… (ii)

Table of Values

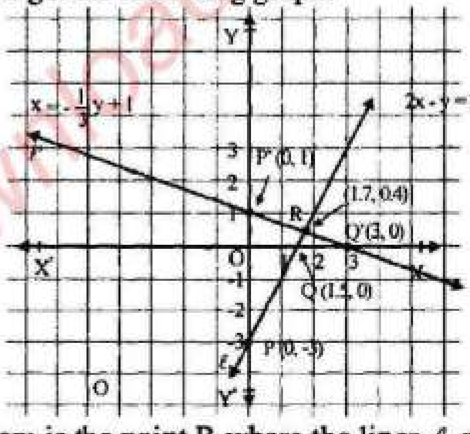
$y = 2x - 3$

x	..., 0	1.5 ..,
y	.... -3	0...

$y = -\frac{1}{3}x + 1$

x	..., 0	3..,
y	.... -1	0...

By plotting the points we get the following graph.



The solution of the system is the point R where the lines ℓ and ℓ' meet at, R(1.7, 04) such that x = 1.7 and y = 0.4.

Example: Solve graphically, the following linear system of two equations in two variables x and y;

$x + 2y = 3$ ......(i) $x - y = 2$ ......(ii)

Solution: The equations (i) and (ii) are represented graphically with the help of their points of intersection with the coordinate axes of the same co-ordinate plane.

The points of intersections of the lines representing equation (i) and (ii) are

given in the following table:

$$y = \frac{-1}{2}x + \frac{3}{2}$$

x	...0	3...
y	... 1.5	0...

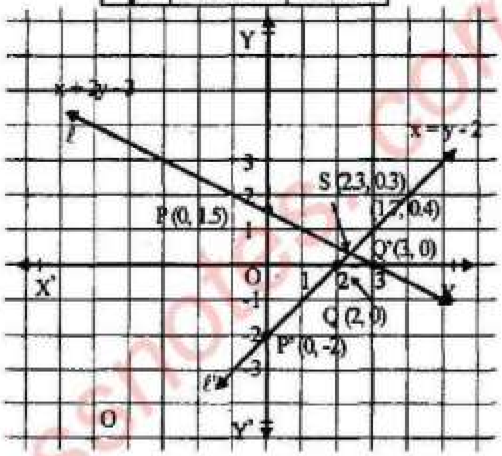
$$y = x - 2$$

x	...0	2....
y	...-2	0...

The points P (0, 1.5) and Q (3, 0) of equation (i) are plotted in the plane and the corresponding line $\ell : x + 2y = 3$ is traced by joining P and Q.

Similarly, the line ℓ' : $x - y = 2$ of (ii) is obtained by plotting the points P' (0, −2) and Q' (2, 0) in the plane and joining them to trace the line ℓ' as below:

The common point S (2.3, 0.3) lies on both the lines, which is the required solution of the system.



Solved Exercise 8.3

Solve the following pair of equations in x and y graphically.

1. $x + y = 0$ and $2x - y + 3 = 0$

Solution: $x + y = 0$

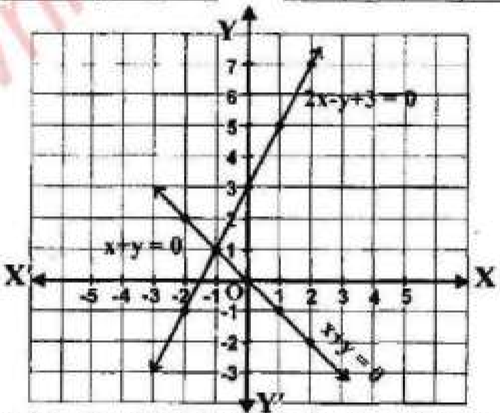
$y = -x$

x	-2	-1	0	1	2
y	2	1	0	-1	-2

$2x - y + 3 = 0$

$y = 2x + 3$

x	-2	-1	0	1	2
y	-1	1	3	5	7



The common point (-1, 1) which is the solution of the system.

2. $x - y + 1 = 0$ and $x - 2y = -1$

Solution: $x - y + 1 = 0$

$y = x + 1$

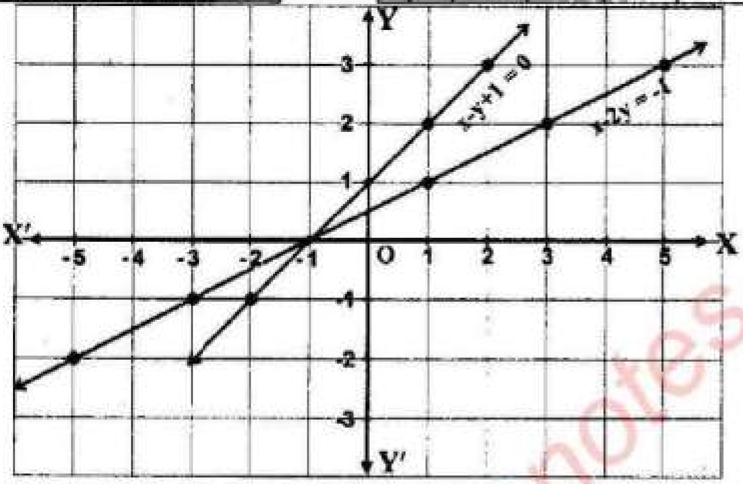
$x - 2y = -1$

$2y = x + 1$

$y = \frac{1}{2}(x+1)$

x	-2	-1	0	1	2
y	-1	0	1	2	3

x	-5	-3	-1	1	3	5
y	-2	-1	0	1	2	3



The common point is (-1, 0) which is the solution of the system.

3. $2x + y = 0$ **and** $x + 2y = 2$

Solution: $2x + y = 0$

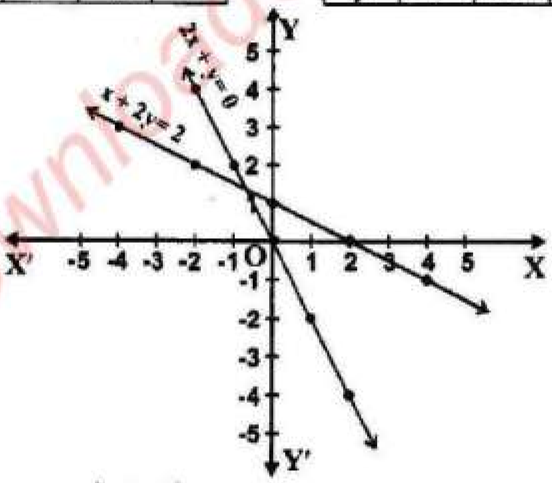
$y = -2x$

x	-2	-1	0	1	2
y	4	2	0	-2	-4

$x + 2y = 2$

$2y = -x + 2 \quad \Rightarrow y = \frac{1}{2}(-x+2)$

x	-4	-2	0	2	4
y	3	2	1	0	-1



The common point is $\left(\frac{-2}{3}, \frac{4}{3}\right)$, which is the solution of the system.

4. $x + y - 1 = 0$ **and** $x - y + 1 = 0$

Solution: $x + y - 1 = 0$

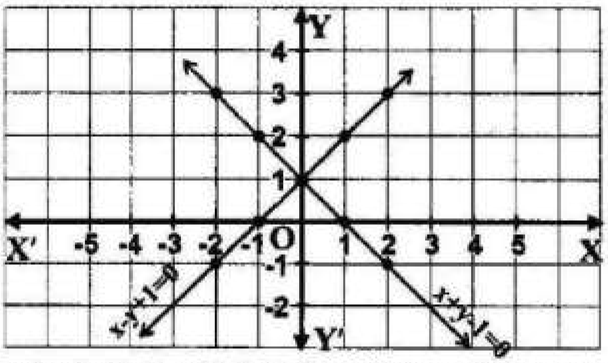
$y = -x + 1$

x	-2	-1	0	1	2
y	3	2	1	0	-1

$x - y + 1 = 0$

$y = x + 1$

x	-2	-1	0	1	2
y	-1	0	1	2	3



The common point is (0,1) which is the solution of the system.

5. $2x + y - 1 = 0$ **and** $x = -y$

Solution: $2x + y - 1 = 0$

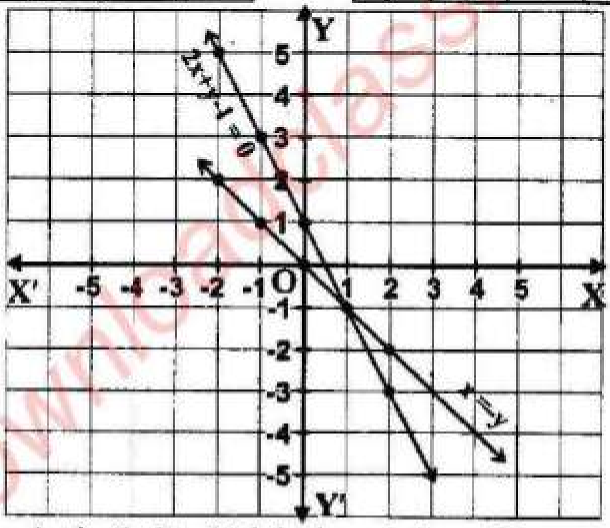
$y = -2x + 1$

x	-2	-1	0	1	2
y	5	3	1	-1	-3

$x = -y$

$y = -x$

x	-2	-1	0	1	2
y	2	1	0	-1	-2



The common point is (1,-1) which is the solution of the system.

Solved Review Exercise 8

1. Choose the correct answer.

(i) If (x-l, y +l) = (0,0), then (x, y) is
(a) (1,-1) (b) (-1,1) (c) (1,1) (d) (-1,-1)

(ii) If (x, 0) = (0, y), then (x, y) is
(a) (0, 1) (b) (1,0) (c) (0,0) (d) (1,1)

(iii) Point (2, -3) lies in quadrant
(a) I (b) II (c) III (d) IV

(iv) Point (-3, -3) lies in quadrant
(a) I (b) II (c) III (d) IV

(v) If $y = 2x + 1$, $x = 2$ then y is

(a) 2 (b) 3 (c) 4 (d) 5

(vi) Which ordered pair satisfies the equation $y = 2x$?

(a) (1,2) (b) (2,1) (c) (2,2) (d) (0,1)

Solution: (i) a (ii) c (iii) d (iv) c (v) d (vi) a

2. Identify the following statements as True or False.

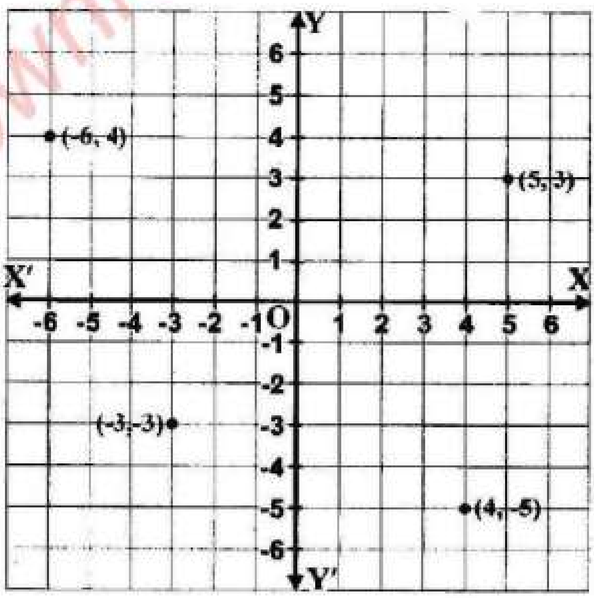
(i) The point O (0, 0) is in quadrant II.
(ii) The point P(2, 0) lies on x-axis.
(iii) The graph of $x = -2$ is a vertical line.
(iv) $3 - y = 0$ is a horizontal line.
(v) The point Q (-1, 2) is in quadrant III.
(vi) The point R (-1, -2) is in quadrant IV.
(vii) $y = x$ is a line on which origin lies.
(viii) The point P (1, 1) lies on the line $x + y = 0$.
(ix) The point S (1,-3) lies in quadrant III.
(x) The point R (0, 1) lies on the x-axis.

Solution: (i) F (ii) T (iii) T (iv) T (v) F
(vi) F (vii) T (viii) F (ix) F (x) F

3. Draw the following points on the graph paper.

(-3, -3), (-6, 4), (4, -5), (5, 3)

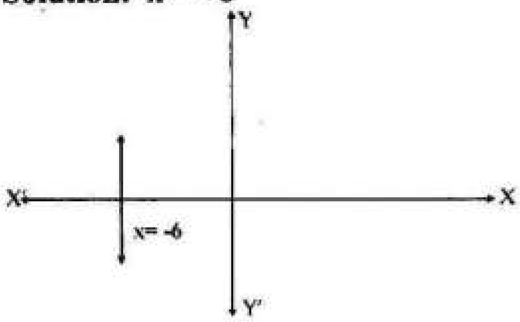
Solution:



4. Draw the graph of the following.

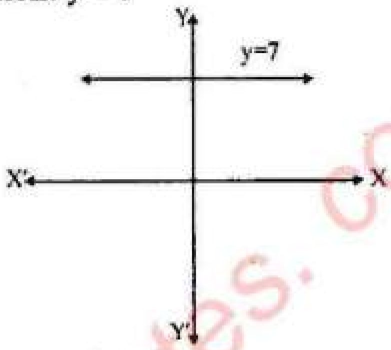
(i) $x = -6$

Solution: $x = -6$



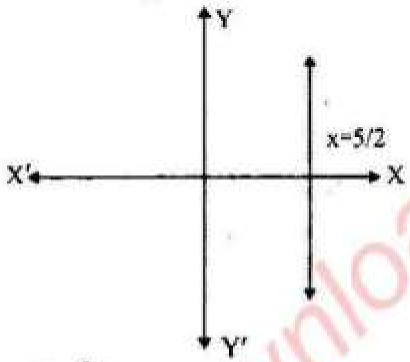
(ii) $y = 7$

Solution: $y = 7$



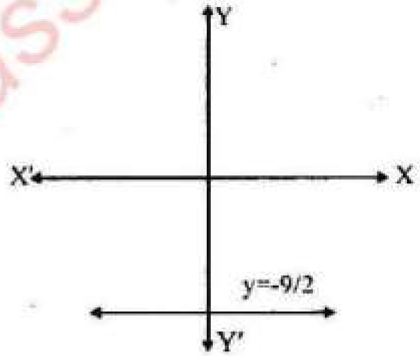
(iii) $x = \frac{5}{2}$

Solution: $x = \frac{5}{2}$



(iv) $y = -\frac{9}{2}$

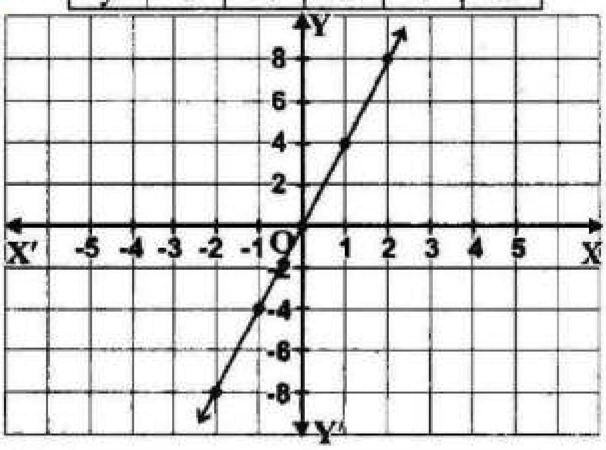
Solution: $y = -\frac{9}{2}$



(v) $y = 4x$

Solution: $y = 4x$

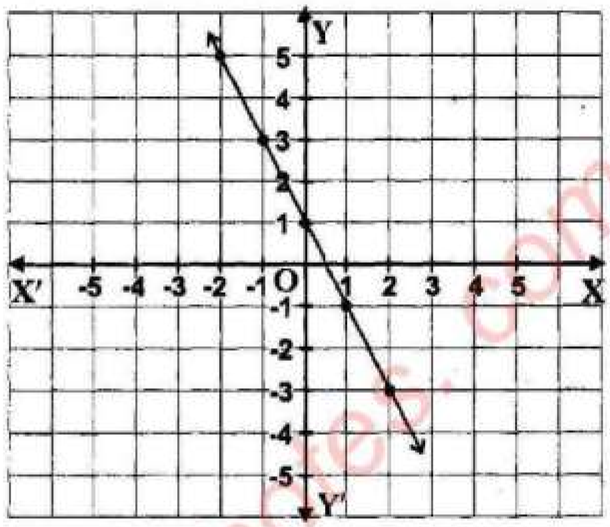
x	-2	-1	0	1	2
y	-8	-4	0	4	8



(vi) $y = -2x + 1$

Solution: $y = -2x + 1$

x	-2	-1	0	1	2
y	5	3	1	-1	-3

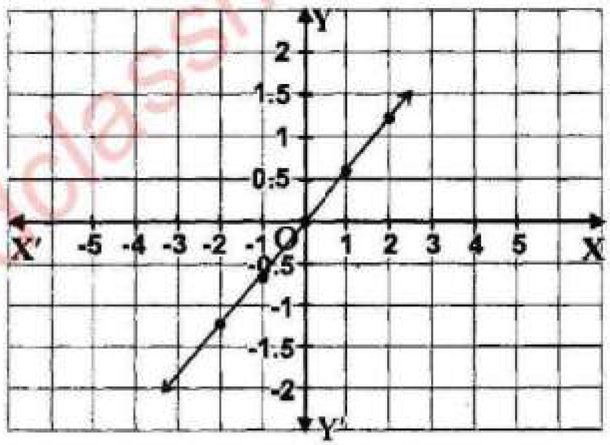


5. Draw the following graph.

(i) $y = 0.62x$

Solution: $y = 0.62x$

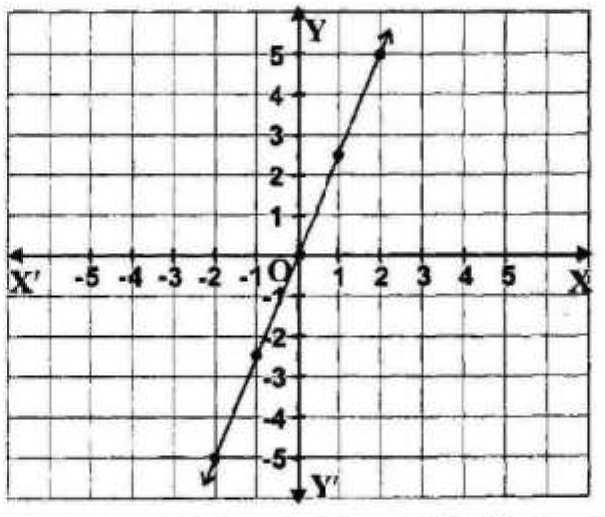
x	-2	-1	0	1	2
y	-1.24	-0.62	0	0.62	1.24



(ii) $y = 2.5x$

Solution: $y = 2.5x$

x	-2	-1	0	1	2
y	-5	-2.5	0	2.5	5



6. Solve the following pair of equations graphically.

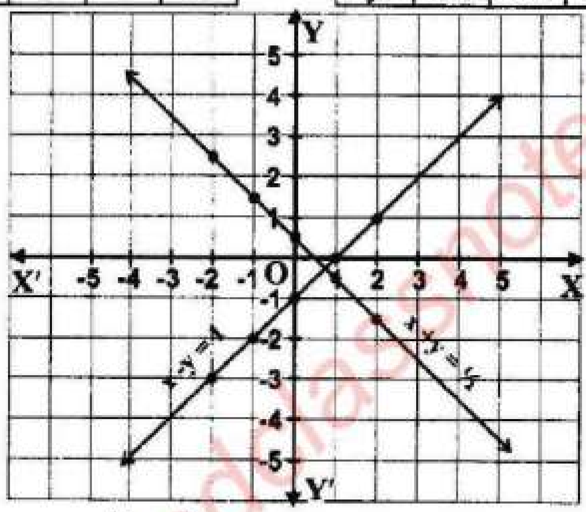
(i) $x - y = 1, \qquad x + y = \frac{1}{2}$

Solution: $x - y = 1$ $\qquad$ $x + y = \frac{1}{2}$

$y = x - 1$ $\qquad$ $y = -x + \frac{1}{2}$

x	-2	-1	0	1	2
y	-3	-2	-1	0	1

x	-2	-1	0	1	2
y	2.5	1.5	0.5	-0.5	-1.5



The common point is $\left(\frac{3}{4}, \frac{-1}{4}\right)$ which is the solution of the system.

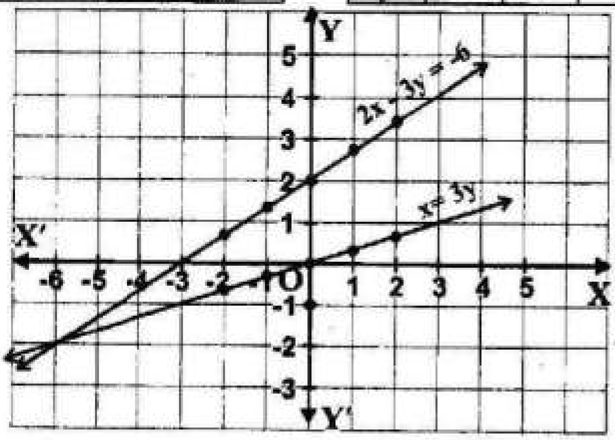
(ii) $x = 3y, 2x - 3y = -6$

Solution: $x = 3y$ $\qquad$ $2x - 3y = -6$

or $y = \frac{1}{3}x$ $\qquad$ $3y = 2x + 6 \quad \Rightarrow y = \frac{1}{3}(2x + 6)$

x	-2	-1	0	1	2
y	-0.7	-0.3	0	0.3	0.7

x	-2	-1	0	1	2
y	0.7	1.3	2	2.7	3.3



The common point is (-6, -2), which is the solution of the system.

(iii) $\frac{1}{3}(x+y)=2,\ \frac{1}{2}(x-y)=-1$

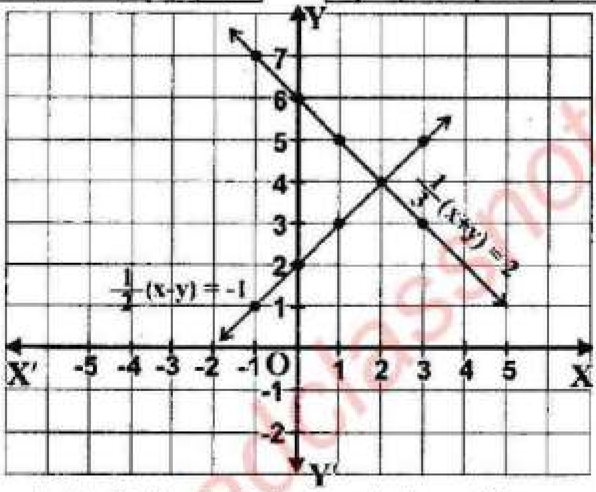
Solution: $\frac{1}{3}(x+y)=2$ $\qquad$ $\frac{1}{2}(x-y)=-1$

or $x+y=6$ $\qquad$ $x-y=-2$

$y=-x+6$ $\qquad$ $y=x+2$

x	-1	0	1	2	3
y	7	6	5	4	3

x	-1	0	1	2	3
y	1	2	3	4	5



The common point is (2, 4) which is the solution of the system.

SUMMARY

- An ordered pair is a pair of elements in which elements are written in specific order.
- The plane formed by two straight lines perpendicular to each other is called Cartesian plane and the lines are called coordinate axes.
- The point of intersection of two coordinate axes is called origin.
- There is a one-to-one correspondence between ordered pair and a point in Cartesian plane and vice versa.
- Cartesian plane is also known as coordinate plane.
- Cartesian plane is divided into four quadrants.
- The x-coordinate of a point is called abcissa and y-coordinate is called ordinate.
- The set of points which lie on the same line are called collinear points.

UNIT 9

INTRODUCTION TO COORDINATE GEOMETRY

Unit Outlines			
9.1	*Introduction*	9.2	*The Distance Formula*
9.3	*Collinear Points*	9.4	*Mid Point Formula*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- define coordinate geometry.
- derive distance formula to calculate distance between two points given in Cartesian plane.
- use distance formula to find distance between two given points.
- define collinear points. Distinguish between collinear and non-collinear points.
- use distance formula to show that given three (or more) points are collinear.
- use distance formula to show that the given three non-collinear points form.
  - ★ an equilateral triangle, ★ an isosceles triangle,
  - ★ a right angled triangle, ★ a scalene triangle.
- use distance formula to show that given four non-collinear points form.
  - ★ a square, ★ a rectangle, ★ a parallelogram.
- recognize the formula to find the midpoint of the line joining two given points.
- apply distance and mid point formulae to solve/verify different standard results related to geometry.

DISTANCE FORMULA

Coordinate Geometry:

The study of geometrical shapes in a plane is called plane geometry. Coordinate geometry is the study of geometrical shapes in the Cartesian plane (coordinate plane).

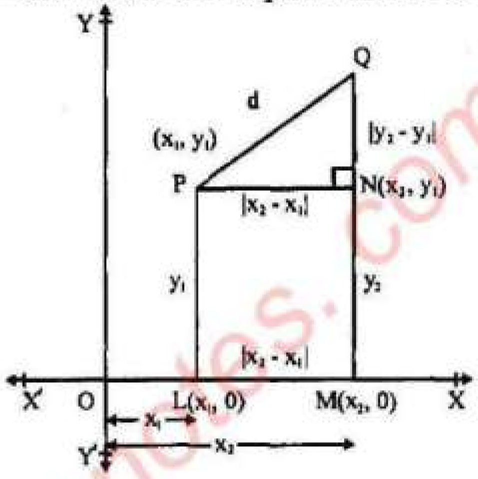
We know that a plane is divided into four quadrants by two perpendicular lines called the axes intersecting at origin. We have also seen that there is one to one correspondence between the points of the plane and the ordered pairs in $R \times R$.

Finding Distance between two points:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the coordinate plane where d is

the length of the line segment PQ. i.e., $|PQ| = d$.

The line segments MQ and LP parallel to y-axis meet x-axis at points M and L respectively with coordinates M $(x_2, 0)$ and L$(x_1, 0)$.



The line-segment PN is parallel to x-axis.

In the right triangle PNQ,

$$|NQ| = |y_2 - y_1|$$

And $|PN| = |x_2 - x_1|$.

Using Pythagoras Theorem

$$(PQ)^2 = (PN)^2 + (QN)^2$$

$$d^2 = |y_2 - y_1| + |x_2 - x_1|^2$$

$$\Rightarrow \quad d = \pm\sqrt{|y_2 - y_1|^2 + |x_2 - x_1|^2}$$

$$d = \sqrt{|y_2 - y_1|^2 + |x_2 - x_1|^2}$$

Use of Distance Formula:

The use of distance formula is explained in the following examples.

Example-1: Using the distance formula, find the distance between the points.

(i) P (l,2) and Q (0,3) (ii) S (-l, 3) and R (3, -2)

(iii) U (0, 2) and V(-3, 0) (iv) P' (l, 1) and Q' (2, 2)

Solution:

(i) $|PQ| = \sqrt{(0-1)^2 + (3-2)^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

(ii) $|SR| = \sqrt{(3-(-1))^2 + (-2-3)^2} = \sqrt{(3+1)^2 + (-5)^2} = \sqrt{16+25} = \sqrt{41}$

(iii) $|UV| = \sqrt{(-3-0)^2 + (0-2)^2} = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$

(iv) $|PQ| = \sqrt{(2-1)^2 + (2-1)^2} = \sqrt{1+1} = \sqrt{2}$

Solved Exercise 9.1

1. Find the distance between the following pairs of points.

(a) A (9, 2), B(7, 2)

Solution: $|AB| = \sqrt{(7-9)^2 + (2-2)^2} = \sqrt{(-2)^2 + (0)^2} = \sqrt{4+0}$
$= \sqrt{4} = 2$

(b) A (2, -6), B(3, -6)

Solution: $|AB| = \sqrt{(3-2)^2 + (-6+6)^2} = \sqrt{(1)^2 + (0)^2} = \sqrt{1+0} = \sqrt{1} = 1$

(c) A (-8, 1), B(6, 1)

Solution: $|AB| = \sqrt{(6-(-8))^2 + (1-1)^2} = \sqrt{(6+8)^2 + (0)^2} = \sqrt{(14)^2 + 0} = \sqrt{196} = 14$

(d) A $(-4, \sqrt{2})$, B$(-4, -3)$

Solution: $|AB| = \sqrt{(-4)-(-4)^2 + \left(-3-\sqrt{2}\right)^2} = \sqrt{(-4+4)^2 + \left(3+\sqrt{2}\right)^2}$

$= \sqrt{(0)^2 + \left(3+\sqrt{2}\right)^2} = \sqrt{\left(3+\sqrt{2}\right)^2} = 3+\sqrt{2}$

(e) A $(3, -11)$, B$(3, -4)$

Solution: $|AB| = \sqrt{(3-3)^2 + (-4-(-11))^2} = \sqrt{(0)^2 + (-4+11)^2} = \sqrt{0+(7)^2} = \sqrt{49} = 7$

(f) A $(0, 0)$, B$(0, -5)$

Solution: $|AB| = \sqrt{(0-0)^2 + (-5-0)^2} = \sqrt{(0)^2 + (-5)^2} = \sqrt{0+25} = \sqrt{25} = 5$

2. Let P be the point on x-axis with x-coordinate a and Q be the point on y-axis with y-coordinate b as given below. Find the distance between P and Q.

(i) $a = 9, b = 7$

Solution: $a = 9, b = 7 \Rightarrow P(a, o) = P(9, 0)$ and $Q(0, b) = Q(0, 7)$

$|PQ| = \sqrt{(0-9)^2 + (7-0)^2} = \sqrt{(-9)^2 + (7)^2} = \sqrt{81+49} = \sqrt{130}$

(ii) $a = 2, b = 3$

Solution: $a = 2, b = 3 \Rightarrow P(a, 0) = P(2, 0)$ and $Q(0, b) = Q(0, 3)$

$|PQ| = \sqrt{(0-2)^2 + (3-0)^2} = \sqrt{(-2)^2 + (3)^2} = \sqrt{4+9} = \sqrt{13}$

(iii) $a = -8, b = 6$

Solution: $a = -8, b = 6 \Rightarrow P(a, 0) = P(-8, 0)$ and $Q(0, b) = Q(0, 6)$

$|PQ| = \sqrt{(0-(-8))^2 + (6-0)^2} = \sqrt{(8)^2 + (6)^2} = \sqrt{64+36} = \sqrt{100} = 10$

(iv) $a = -2, b = -3$

Solution: $a = -2, b = -3 \Rightarrow P(a, 0) = P(-2, 0)$ and $Q(0, b) = Q(0, -3)$

$|PQ| = \sqrt{(0-(-2))^2 + (-3-0)^2} = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

(v) $a = \sqrt{2}, b = 1$

Solution: $a = \sqrt{2}, b = 1 \Rightarrow P(a, 0) = P(\sqrt{2}, 0)$ and $Q(0, b) = Q(0, 1)$

$|PQ| = \sqrt{\left(0-\sqrt{2}\right)^2 + (1-0)^2} = \sqrt{\sqrt{2}^2 + (1)^2}$

$= \sqrt{2+1} = \sqrt{3}$

(vi) $a = -9, b = -4$

Solution: $a = -9, b = -4 \Rightarrow P(a, 0) = P(-9, 0)$ and $Q(0, b) = Q(0, -4)$

$|PQ| = \sqrt{(0-(-9))^2 + (-4-0)^2} = \sqrt{(9)^2 + (-4)^2}$

$= \sqrt{81+16} = \sqrt{97}$

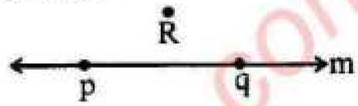
COLLINEAR POINTS

Collinear or Non-collinear Points in the Plane:

Two or more than two points which lie on the same straight line are called collinear points with respect to that line; otherwise they are called non-collinear.

Let 'm' be a line, then all the points on line m are collinear.

In the given figure the points P and Q are collinear with respect to the line m and the points P and R are not collinear with respect to it.



Use of Distance Formula to show the Collinearity of Three or more Points in the Plane:

Let P, Q and R be three points in the plane. They are called collinear, if $|PQ| + |QR| = |PR|$, otherwise they are non-collinear.

Example: Using distance formula show that the points

(i) P (-2, -1), Q (0, 3) and R (1, 5) are collinear.

(ii) The above points P, Q, R and S (1, -1) are not collinear.

Solution: (i) By using the distance formula, we find.

$$|PQ| = \sqrt{(0+2)^2 + (3+1)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$|QR| = \sqrt{(1-0)^2 + (5-3)^2} = \sqrt{1+4} = \sqrt{5}$$

$$|PR| = \sqrt{(1+2)^2 + (5+1)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

Since $|PQ| + |QR| = 2\sqrt{5} + \sqrt{5} = 3\sqrt{5} = |PR|$

Therefore, the points P, Q and R are collinear

(ii) $|PS| = \sqrt{(-2-1)^2 + (-1+1)^2} = \sqrt{(-3)^2 + 0} = 3$

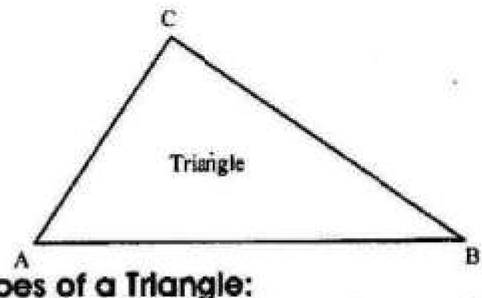
Since $|QS| = \sqrt{(1-0)^2 + (-1-3)^2} = \sqrt{1+16} = \sqrt{17}$

And $|PQ| + |QS| \neq |PS|$,

Therefore the points P, Q and S are not collinear and hence, the points P, Q, R and S are also not collinear.

A closed figure in a plane obtained by joining three non-collinear points is called a **triangle**.

In the triangle ABC the non-collinear points A, B and C are the three vertices of the triangle ABC. The line segments AB, BC and CA are called **sides of the triangle.**



Use of Distance Formula to Different Shapes of a Triangle:

We expand the idea of a triangle to its different kinds depending on the length of the three sides of the triangle as:

(i) Equilateral triangle (ii) Isosceles triangle
(iii) Right angled triangle (iv) Scalene triangle

(i) Equilateral Triangle:

If the lengths of all the three sides of a triangle are same, then the triangle is called an equilateral triangle.

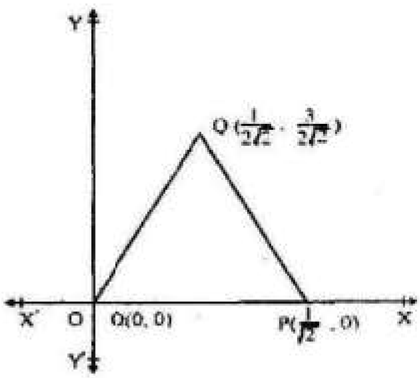
Example: The triangle OPQ is an equilateral triangle since the points O (0,0), $P\left(\frac{1}{\sqrt{2}},0\right)$ and $Q\left(\frac{1}{2\sqrt{2}},\frac{\sqrt{3}}{2\sqrt{2}}\right)$ are not collinear.

Where $|OP| = \frac{1}{\sqrt{2}}$

$$|QO| = \sqrt{\left(0-\frac{1}{2\sqrt{2}}\right)^2+\left(0-\frac{\sqrt{3}}{2\sqrt{2}}\right)^2} = \sqrt{\frac{1}{8}+\frac{3}{8}} = \sqrt{\frac{4}{8}} = \sqrt{\frac{1}{2}}$$

$$|PQ| = \sqrt{\left(\frac{1}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{\sqrt{3}}{2\sqrt{2}}-0\right)^2}$$

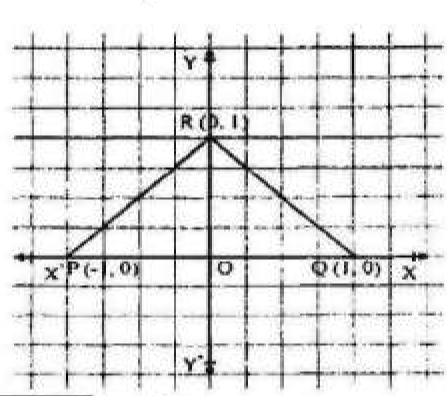
$$= \sqrt{\left(\frac{1-2}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2} = \sqrt{\frac{1}{8}+\frac{3}{8}} = \sqrt{\frac{4}{8}} = \sqrt{\frac{1}{2}}$$



i.e., $|OP| = |QO| = |PQ| = \frac{1}{\sqrt{2}}$ a real number and the points O (0,0), $Q\left(\frac{1}{2\sqrt{2}},\frac{\sqrt{3}}{2\sqrt{2}}\right)$ and $P\left(\frac{1}{\sqrt{2}}\right)$ are not collinear. Hence the triangle OPQ is equilateral.

(ii) An Isosceles Triangle:

An isosceles triangle PQR is a triangle which has two of its sides with equal length while the third side has a different length.



Example:

The triangle PQR is an isosceles triangle as for the non-collinear points P (-l, 0), Q (l, 0) and R (0, 1) shown in the following figure.

$$|PO| = \sqrt{(1-(-1))^2+(0-0)^2} = \sqrt{(1+1)^2+0} = \sqrt{4} = 2$$

$$|QR| = \sqrt{(0-1)^2+(1-0)^2} = \sqrt{(-1)^2+1^2} = \sqrt{1+1} = \sqrt{2}$$

$|PR| = \sqrt{(0-(-1))^2+(1-0)^2} = \sqrt{(-1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$

Since $|QR| = |PR| = \sqrt{2}$ and $|PQ| = 2 \neq \sqrt{2}$ so the non-collinear points P, Q, R form an isosceles triangle PQR.

(iii) Right Angle Triangle:

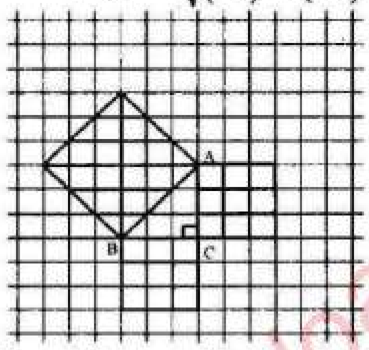
A triangle in which one of the angles has measure equal to 90° is called a right angle triangle.

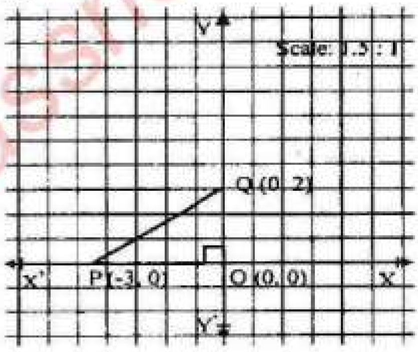
Example: Let O (0, 0), P (-3, 0) and 0 (0, 2) be three non-collinear points. Verify that triangle OPQ is right-angled.

Solution: $|OQ| = \sqrt{(0-0)^2+(2-0)^2} = \sqrt{2^2} = 2$

$|OP| = \sqrt{(-3)^2+0^2} = \sqrt{9} = 3$

$|PQ| = \sqrt{(-3)^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$





Here 1.5 square block = 1 unit length

Now $|OQ|^2 + |OP|^2 = (2)^2 + (3)^2 = 13$ and $|PQ| = 13$

Since $|OQ|^2 + |OP|^2 = |PQ|^2$, therefore $\angle POQ = 90°$

Hence the given non-collinear points form a right triangle.

(iv) Scalene Triangle:

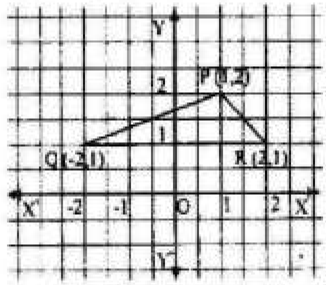
A triangle is called a scalene triangle if measures of all the three sides are different.

Example: Show that the points P(l, 2), Q(-2, 1) and R(2, 1) in the plane form a scalene triangle.

Solution: $|PQ| = \sqrt{(-2-1)^2+(1-2)^2} = \sqrt{(-3)^2+(-1)^2} = \sqrt{9+1} = \sqrt{10}$

$|QR| = \sqrt{(2+2)^2+(1-1)^2} = \sqrt{4^2+0^2} = \sqrt{4^2} = 4$

$|PR| = \sqrt{(2-1)^2+(1-2)^2} = \sqrt{1^1+(-1)^2} = \sqrt{1^2+1^2} = \sqrt{2}$



Hence $|PQ| = \sqrt{10}$, $|QR| = 4$ and $|PR| = \sqrt{2}$

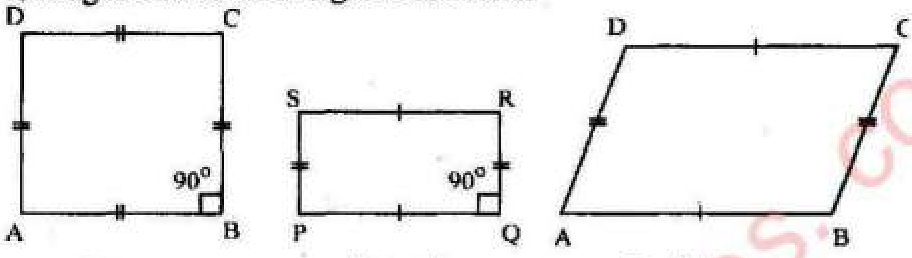
The points P, Q and R are non-collinear. Since,

$|PQ| + |QR| > |PR|$

Thus the given points form a scalene triangle.

Use of distance formula to show that four non-collinear points form a square, a rectangle and a parallelogram:

We recognize these three figures as below.



Square Rectangle Parallelogram

(a) Using Distance Formula to show that given four Non-Collinear Points form a Square:

A square is a closed figure in the plane formed by four non-collinear points such that lengths of all sides are equal and measure of each angle is 90°.

Example: If A (2, 2), B (2, -2), C (-2, -2) and D (-2, 2) be four non-collinear points in the plane, then verify that they form a square ABCD.

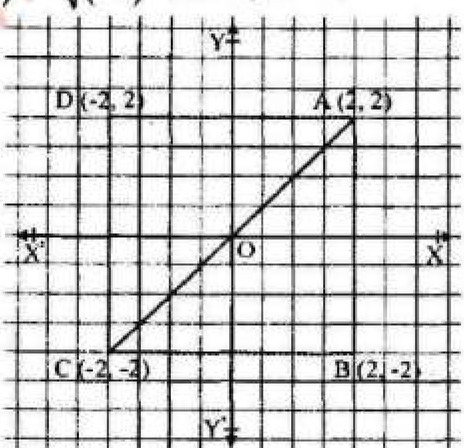
Solution:

$|AB| = \sqrt{(2-2)^2+(-2-2)^2} = \sqrt{0^2+(-4)^2} = \sqrt{16} = 4$

$|BC| = \sqrt{(-2-2)^2+(-2+2)^2} = \sqrt{(-4)^2-0^2} = \sqrt{16} = 4$

$|CD| = \sqrt{(-2-(-2))^2+(2-(-2))^2} = \sqrt{(-2+2)^2+(2+2)^2} = \sqrt{0+16} = \sqrt{16} = 4$

$|DA| = \sqrt{(2+2)^2+(2-2)^2} = \sqrt{(+4)^2+0} = \sqrt{16} = 4$



Hence $|AB| = |BC| = |CD| = |DA| = 4$.

Also $|AC| = \sqrt{(-2-2)^2+(-2-2)^2} = \sqrt{(-4)^2+(-4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

Now $|AB|^2 + |BC|^2 = (4)^2 + (4)^2 = 32$, and $|AC|^2 = (4\sqrt{2})^2 = 32$

Since $|AB|^2 + |BC|^2 = |AC|^2$, therefore $\angle ABC = 90°$
Hence the given four non collinear points form a square.

(b) Using Distance Formula to show that given four Non-Collinear Points form a Rectangle:

A figure formed in the plane by four non-collinear points is called a rectangle if,
(i) its opposite sides are equal in length;
(ii) the angle at each vertex is of measure 90°.

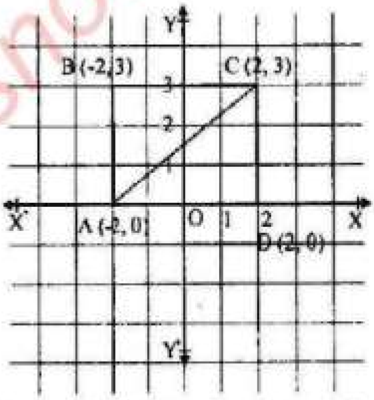
Example: Show that the points A (-2, 0), B (-2, 3), C (2, 3) and D (2, 0) form a rectangle.

Solution: $|AB| = \sqrt{(-2+2)^2+(3-0)^2} = \sqrt{0+9} = \sqrt{9} = 3$

$|DC| = \sqrt{(2-2)^2+(3-0)^2} = \sqrt{0+9} = \sqrt{9} = 3$

$|AD| = \sqrt{(2+2)^2+(0-0)^2} = \sqrt{16+0} = 4$

$|BC| = \sqrt{(2+2)^2+(3-3)^2} = \sqrt{16+0} = \sqrt{16} = 4$



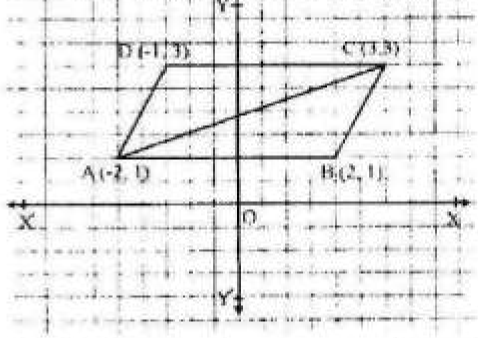
Since $|AB| = |DC| = 3$ and $|AD| = |BC| = 4$
Therefore, opposite sides are equal.
Also $|AC| = \sqrt{(2+2)^2+(3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5$
Now $|AD|^2 + |DC|^2 = (4)^2 + (3)^2 = 25$
And $= |AC|^2 = (5)^2 = 25$
Since $|AD|^2 + |DC|^2 = |AC|^2$
Therefore $\angle ADC = 90°$
Hence the given points form a rectangle.

(c) Use of Distance Formula to show that given four Non-Collinear Points Form a Parallelogram:

Definition: A figure formed by four non-collinear points in the plane is called a parallelogram if:
(i) its opposite sides are of equal length (ii) its opposite sides are parallel
(iii) measure of none of the angles is 90°.

Example: Show that the points A(-2,1), B(2,1), C(3,3) and D(-1,3) form a parallelogram.

Solution:



By distance formula,

$|AB| = \sqrt{(2+2)^2+(1-1)^2} = \sqrt{4^2+0} = \sqrt{16} = 4$

$|CD| = \sqrt{(3+1)^2+(3-3)^2} = \sqrt{4^2+0} = \sqrt{16} = 4$

$|AD| = \sqrt{(-1+2)^2+(3-1)^2} = \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5}$

$|BC| = \sqrt{(3-2)^2+(3-1)^2} = \sqrt{1^2+2^2} = \sqrt{5}$

Since $|AB| = |CD| = 4$ and $|AD| = |BC| = \sqrt{5}$

So opposite sides of the quadrilateral ABCD are equal.

Also $|AC| = \sqrt{(3+2)^2+(3-1)^2} = \sqrt{(5)^2+2^2} = \sqrt{25+4} = \sqrt{29}$

Now $|AB|^2 + |BC|^2 = 16 + 5 = 21$ and $|AC|^2 = 29$

Since in triangle ABC, $|AB|^2 + |BC|^2 \neq |AC|^2$

Therefore measure of angle at $B \neq 90°$.

Hence the given points form a parallelogram.

Solved Exercise 9.2

1. Show whether the points with vertices (5, -2), (5, 4) and (-4, 1) are vertices of an equilateral triangle or an isosceles triangle?

Solution: Let the vertices are A (5, - 2), B (5, 4), C (-4, 1)

$$|AB| = \sqrt{(5-5)^2+(4-(-2))^2} = \sqrt{(0)^2+(4+2)^2}$$

$$= \sqrt{0+(6)^2} = \sqrt{0+36} = \sqrt{36} = 6$$

$$|BC| = \sqrt{(-4-5)^2+(1-4)^2} = \sqrt{(-9)^2+(-3)^2} = \sqrt{81+9} = \sqrt{90} = 3\sqrt{10}$$

$$|AC| = \sqrt{(-4-5^2)+[1-(-2)]^2} = \sqrt{(-9)^2+(3)^2} = \sqrt{81+9} = \sqrt{90} = 3\sqrt{10}$$

Since $|BC| = |AC| = 3\sqrt{10}$ and $|AB| = 6 \neq 3\sqrt{10}$.

So the given points form an isosceles triangle.

2. Show whether or not the points with vertices (-1, 1), (5, 4), (2, -2) and (-4, 1) form a square?

Solution: Let the vertices are A (−1,1), B (5, 4), C (2, −2) and D (−4, 1).

$$|AB| = \sqrt{(5-(-1))^2+(4-1)^2} = \sqrt{(6)^2+(3)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$$

$$|BC| = \sqrt{(2-5)^2+(-2-4)^2} = \sqrt{(-3)^2+(-6)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

$$|CD| = \sqrt{(-4-2)^2+[1-(-2)]^2} = \sqrt{(-6)^2+(3)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$$

$$|DA| = \sqrt{[-4-(-1)]^2+(1-1)^2} = \sqrt{(-3)^2+(0)^2} = \sqrt{9+0} = \sqrt{9} = 3$$

Since $|BC| = |BC| = |CD| = |DA| \neq 3\sqrt{5}$

So the given points do not form a square.

3. Show whether or not the points with coordinates (1, 3), (4, 2) and (-2, 6) are vertices of a right triangle?

Solution: Let the vertices are A (1, 3), B (4, 2), and C (-2, 6).

$$|AB|=\sqrt{(4-1)^2+(2-3)^2}=\sqrt{(3)^2+(-1)^2}=\sqrt{9+1}=\sqrt{10}$$

$$|BC|=\sqrt{(-2-4)^2+(6-2)^2}=\sqrt{(-6)^2+(4)^2}=\sqrt{36+16}=\sqrt{52}=2\sqrt{13}$$

$$|AC|=\sqrt{(-2-1)^2+(6-3)^2}=\sqrt{(-3)^2+(3)^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$$

As $|AB|^2+|BC|^2=|AC|^2 \Rightarrow \left(\sqrt{10}\right)^2+\left(\sqrt{52}\right)^2=\left(\sqrt{18}\right)^2$

$\Rightarrow 10 + 52 = 18$

$\Rightarrow 62 \neq 18$

Or $|AB|^2+|AC|^2=|BC|^2 \Rightarrow \left(\sqrt{10}\right)^2+\left(\sqrt{18}\right)^2=\left(\sqrt{52}\right)^2$

$\Rightarrow 10 + 18 = 52$

$\Rightarrow 28 \neq 52$

So the given points do not form a right angle triangle.

4. Use the distance formula to prove whether or not the points (1, 1), (-2, -8) and (4, 10) lie on a straight line.

Solution: Let the vertices are A (1, 1), B (-2, -8) and C (4, 10)

$$|AB|=\sqrt{(-2-1)^2+(-8-1)^2}=\sqrt{(-3)^2+(-9)^2}=\sqrt{9+81}=\sqrt{90}=3\sqrt{10}$$

$$|BC|=\sqrt{[4-(-2)]^2+[10-(-8)]^2}=\sqrt{(4+2)^2+(10+8)^2}$$

$$=\sqrt{(6)^2+(18)^2}=\sqrt{36+324}=\sqrt{360}=6\sqrt{10}$$

$$|AC|=\sqrt{(4-1)^2+(10-1)^2}=\sqrt{(3)^2+(9)^2}=\sqrt{9+81}=\sqrt{90}=3\sqrt{10}$$

Since $|AB|+|AC|=|BC|$

$3\sqrt{10}+3\sqrt{10}=6\sqrt{10}$

$6\sqrt{10}=6\sqrt{10}$

Therefore, the points A, B and C are collinear or the given points lie on a straight line.

5. Find k, given that the point (2, k) is equidistant from (3, 7) and (9, 1).

Solution: Let the vertices are A (2, k), B (3, 7), and C (9, 1).

As the point A is equidistant from B and C, So

$$|AB| = |AC|$$

$$\sqrt{(3-2)^2+(7-k)^2}=\sqrt{(9-2)^2+(1-k)^2}$$

$$\sqrt{(1)^2+(7-k)^2}=\sqrt{(7)^2+(1-k)^2}$$

Squaring both sides, we get

$$1+49-14k+k^2=49+1-2k+k^2$$
$$50-14k=50-2K$$
$$-14k+2k=-50+50 \qquad \Rightarrow -12k=0$$
$$\Rightarrow \qquad k=0$$

6. Use distance formula to verify that the points A(0, 7), B(3, -5), C(-2, 15) are collinear.

Solution: Let the vertices are A (0, 7), B (3, -5), and C (-2, 15)

$$|AB|=\sqrt{(3-0)^2+(-5-7)^2}=\sqrt{(3)^2+(-12)^2}=\sqrt{9+144}=\sqrt{153}=3\sqrt{17}$$
$$|BC|=\sqrt{(-2-3)^2+(15+5)^2}=\sqrt{(-5)^2+(20)^2}=\sqrt{25+400}=\sqrt{425}=5\sqrt{17}$$
$$|AC|=\sqrt{(-2-0)^2+(15-7)^2}=\sqrt{(-2)^2+(8)^2}=\sqrt{4+64}=\sqrt{68}=2\sqrt{17}$$

Since $|AB|+|AC|=|BC|$

$$3\sqrt{17}+2\sqrt{17}=5\sqrt{17}$$
$$\Rightarrow \qquad 5\sqrt{17}=5\sqrt{17}$$

Therefore, the points A, B and C are collinear.

7. Verify whether or not the points O(0, 0), A ($\sqrt{3}$, 1), B($\sqrt{3}$, -1) are the vertices of an equilateral triangle.

Solution: Let the vertices are O (0, 0), A $(\sqrt{3},1)$, B $(\sqrt{3},-1)$

$$|OA| = \sqrt{\left(\sqrt{3}-0\right)^2+(1-0)^2}=\sqrt{\left(\sqrt{3}\right)^2+(1)^2}=\sqrt{3+1}=\sqrt{4}=2$$
$$|OB| = \sqrt{\left(\sqrt{3}-0\right)^2+(-1-0)^2}=\sqrt{3+1}=\sqrt{4}=2$$
$$|AB| = \sqrt{\left(\sqrt{3}-\sqrt{3}\right)^2+(-1-1)^2}=\sqrt{(0)^2+(-2)^2}=\sqrt{0+4}=\sqrt{4}=2$$

Since $|OA|=|OB|=|AB|=2$

Therefore the given vertices form an equilateral triangle.

8. Show that the points A (-6, -5), B(5, -5), C(5, -8) and D (-6, -8) are vertices of a rectangle. Find the lengths of its diagonals. Are they equal?

Solution: Let the vertices are A (-6, - 5), B (5, -5), C (5, -8) and D (-6, -8).

$$|AB| = \sqrt{[5-(-6)]^2+[-5-(-5)]^2}=\sqrt{(5+6)^2+(-5+5)^2}$$
$$= \sqrt{(11)^2+(0)^2}=\sqrt{121}=11$$
$$|BC|=\sqrt{(5-5)^2+[-8-(-5)]^2}=\sqrt{(0)^2+(-8+5)^2}$$

$$= \sqrt{(0)^2+(-3)^2} = \sqrt{0+9} = \sqrt{9} = 3$$

$$|CD| = \sqrt{(-6-5)^2+[-8-(-8)]^2} = \sqrt{(11)^2+(0)^2} = \sqrt{121+0} = \sqrt{121} = 11$$

$$|DA| = \sqrt{[-6-(-6)]^2+[-8-(-5)]^2} = \sqrt{(-6+6)^2+(-8+5)^2}$$

$$= \sqrt{(0)^2+(-3)^2} = \sqrt{0+9} = \sqrt{9} = 3$$

Since $|AB| = |CD| = 11$ and $|BC| = |DA| = 3$

Therefore, opposite sides are equal.

Now the diagonals are

$$|AC| = \sqrt{[5-(-6)]^2+[-8-(-5)]^2} = \sqrt{(5+6)^2+(-8+5)^2}$$

$$= \sqrt{(11)^2+(-3)^2} = \sqrt{121+9} = \sqrt{130}$$

$$|BD| = \sqrt{[(-6-5)^2+(-8-(-5))]^2} = \sqrt{(-11)^2+(-3)^2} = \sqrt{121+9} = \sqrt{130}$$

As $|AC| = |BD| = \sqrt{130}$

Hence the diagonals are equal.

Now $|DA|^2 + |CD|^2 = |AC|^2$

$$(3)^2+(11)^2 = \left(\sqrt{130}\right)^2$$

$$9+121 = 130$$

$$130 = 130$$

Therefore, $\angle ADC = 90^\circ$

Hence the given points form a rectangle.

9. Show that the points M (-1, 4), N(-5, 3), P(1, - 3) and Q(5, -2) are the vertices of a parallelogram.

Solution: Let the vertices are M (–1, 4), N (–5, 3), P (1, –3) and Q (5, –2)

$$|MN| = \sqrt{[-5-(-1)]^2+(3-4)^2} = \sqrt{(-5+1)^2+(3-4)^2}$$

$$= \sqrt{(-4)^2+(-1)^2} = \sqrt{16+1} = \sqrt{17}$$

$$|NP| = \sqrt{[1-(-5)]^2+(-3-3)^2} = \sqrt{(1+5)^2+(-3-3)^2}$$

$$= \sqrt{(6)^2+(-6)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$$

$$|PQ| = \sqrt{(5-1)^2+[-2-(-3)]^2} = \sqrt{(5-1)^2+(-2+3)^2}$$

$$= \sqrt{(4)^2+(1)^2} = \sqrt{16+1} = \sqrt{17}$$

$$|QM| = \sqrt{[5-(-1)]^2+(-2-4)^2} = \sqrt{(5+1)^2+(-2-4)^2}$$

$$=\sqrt{(6)^2+(-6)^2} = \sqrt{36+36}$$
$$=\sqrt{72} \quad = 6\sqrt{2}$$

Since $|MN| = |PQ| = \sqrt{17}$ and $|NP| = |QM| = 6\sqrt{2}$

So opposite sides of the quadrilateral MNPQ are equal.

Also $|MP| = \sqrt{[1-(-1)]^2+(-3-4)^2} = \sqrt{(1+1)^2+(-3-4)^2}$
$$= \sqrt{(2)^2+(-7)^2} = \sqrt{4+49}$$
$$= \sqrt{53}$$

Now $|MN|^2 + |NP|^2 = \left(\sqrt{17}\right)^2 + \left(\sqrt{72}\right)^2 = 17+72 = 89$ and $|MP|^2 = 53$

Since in triangle MNP, $|MN|^2 + |NP|^2 \neq |MP|^2$

Therefore, measure of angle at $N \neq 90^\circ$

Hence the given points form a parallelogram.

10. Find the length of the diameter of the circle having centre at C(-3, 6) and passing through P(l, 3).

Solution: Radius $= r = |CP| = \sqrt{(-3-1)^2+(6-3)^2}$
$$= \sqrt{(-4)^2+(3)^2} = \sqrt{16+9}$$
$$= \sqrt{25} = 5$$

Diameter $= d = 2r = 2(5) = 10$

MID POINT FORMULA

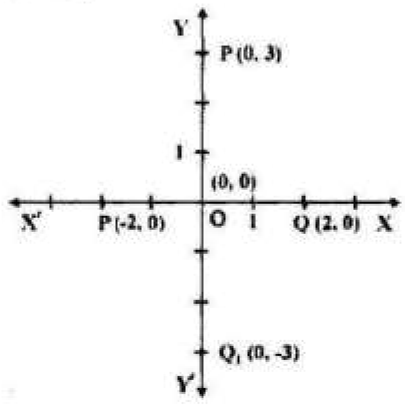
Recognition of the Mid-Point:

Let P (–2, 0) and Q (2, 0) are two points on the x-axis. Then the origin O (0, 0) is the mid point of P and Q.

Since $|OP| = |OQ| = 2$ and the point P, O and Q are collinear.

Similarly the origin is the mid-point of the points P_1 (0, 3) and Q_1 (0, –3).

Since $|OP_1| = |OQ_1| = 3$ and the points P_1, O and Q_1 are collinear.

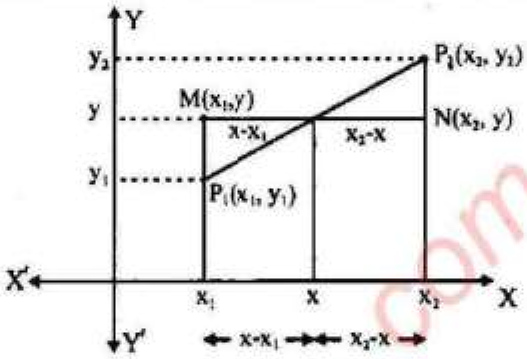


Recognition of the Mid-Point Formula for any two Points in the Plane:

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points in the plane and $R(x, y)$ be a mid-point of points P_1 and P_2 on the line-segment P_1P_2 as shown in the figure below.

If line-segment MN, parallel to x-axis, has its mid point $R(x, y)$, then,

$$x_2 - x = x - x_1$$
$$x + x = x_1 + x_2$$
$$\Rightarrow \quad 2x = x_1 + x_2$$
$$\Rightarrow \quad x = \frac{x_1 + x_2}{2}$$



Similarly, $y = \frac{y_1 + y_2}{2}$

Thus the point $R(x,y) = R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

is the mid-point of the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Verification of the Mid-Point Formula:

$$|P_1R| = \sqrt{\left(\frac{x_1+x_2}{2} - x_1\right)^2 + \left(\frac{y_1+y_2}{2} - y_1\right)^2} = \sqrt{\left(\frac{x_2-x_1}{2}\right)^2 + \left(\frac{y_2-y_1}{2}\right)^2}$$
$$= \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \frac{1}{2}|P_1P_2|$$

And $\quad |P_2R| = \sqrt{\left(\frac{x_1+x_2}{2} - x_2\right)^2 + \left(\frac{y_1+y_2}{2} - y_2\right)^2}$

$$= \sqrt{\left(\frac{x_1+x_2-2x_2}{2}\right)^2 + \left(\frac{y_1+y_2-2y_2}{2}\right)^2} = \sqrt{\left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2}$$
$$= \sqrt{\left(\frac{x_2-x_1}{2}\right)^2 + \left(\frac{y_2-y_1}{2}\right)^2} = \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$
$$\Rightarrow |P_2R| = |P_1R| = \frac{1}{2}|P_1P_2|$$

Thus it verifies that $R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ is the mid-point of the line segment P_1P_2 which lies on the line segment. Since $|P_1R| + |P_2R| = |P_1P_2|$

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, then the mid-point $R(x, y)$ of the line segment PQ is

$$R(x, y) = R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Example-1: Find the mid-point of the line segment joining A(2, 5) and B(-l, 1).
Solution: If R(x, y) is the desired mid point then,

$$x = \frac{2-1}{2} = \frac{1}{2} \qquad \text{and} \qquad y = \frac{5+1}{2} = \frac{6}{2} = 3$$

Hence $R(x, y) = R\left(\frac{1}{2}, 3\right)$

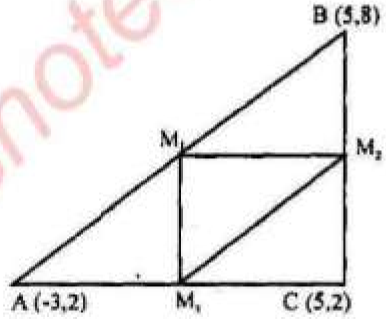
Example-2: Let P(2, 3) and Q(x, y) be two points in the plane such that R(l, -1) is the mid-point of the points P and Q. Find x and y.

Solution: Since R (l, −1) is the mid point of P (2, 3) and Q(x, y) then,

$$1=\frac{x+2}{2} \qquad \text{and} \qquad -1=\frac{y+3}{2}$$
$$2=x+2 \qquad\qquad -2=y+3$$
$$x=0 \qquad\qquad y=-5$$

Example-3: Let ABC be a triangle as shown below. If M_1, M_2 and M_3 are the middle points of the line-segments AB, BC and CA respectively, find the coordinates of M_1, M_2 and M_3. Also determine the type of the triangle $M_1\,M_2\,M_3$.

Solution:



Mid point of AB = $M_1\left(\frac{-3+5}{2},\frac{2+8}{2}\right)=M_1(1,5)$

Mid point of BC = $M_2\left(\frac{5+5}{2},\frac{8+2}{2}\right)=M_2(5,5)$

Mid point of AC = $M_3\left(\frac{5-3}{2},\frac{2+2}{2}\right)=M_3\ (1,2)$

The triangle $M_1M_2M_3$ has sides with length

$$|M_1M_2|=\sqrt{(5-1)^2+(5-5)^2}=\sqrt{4^2+0}=4$$

$$|M_2M_3|=\sqrt{(1-5)^2+(2-5)^2}=\sqrt{(-4^2)+(-3)^2}=\sqrt{16+9}=\sqrt{25}=5$$

$$|M_1M_3|=\sqrt{(1-1)^2+(2-5)^2}=\sqrt{0^2+(-3)^2}=3$$

All the lengths of the three sides are different. Hence the triangle $M_1M_2M_3$ is a scalene triangle.

Example-4: Let O (0, 0), A (3, 0) and B(3, 5) be three points in the plane. If M_1 is the mid point of AB and M_2 of OB, then show that $|M_1M_2|=\frac{1}{2}|OA|$.

Solution: By the distance formula the distance

$$|OA|=\sqrt{(3-0)^2+(0-0)^2}=\sqrt{3^2}=3$$

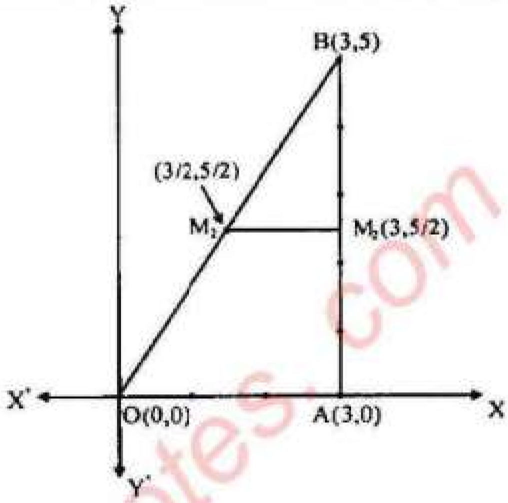
The mid-point of AB is $M_1=M_1\left(\frac{3+3}{2},\frac{5}{2}\right)=\left(3,\frac{5}{2}\right)$.

The mid-point of OB is $M_2=M_2\left(\frac{3+0}{2},\frac{5+0}{2}\right)=\left(\frac{3}{2},\frac{5}{2}\right)$.

Hence

$$|M_1M_2|=\sqrt{\left(\frac{3}{2}-3\right)^2+\left(\frac{5}{2}-\frac{5}{2}\right)^2}$$

$$=\sqrt{\left(\frac{-3}{2}\right)^2+0}=\sqrt{\frac{9}{4}+0}=\frac{3}{2}=\frac{1}{2}|OA|$$



Let P (x_1, y_1) and Q (x_2, y_2) be any two points and their midpoint be

$M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$. Then M

(i) is at equal distance from P and Q
i.e., $|PM| = |MQ|$

(ii) is an interior point of the line segment PQ.

(iii) every point R in the plane at equal distance from P and Q is not their mid point.

For example, the point R (0,1) is at equal distance from P(−3, 0) and Q(3, 0) but is not their mid-point

i.e., $|RQ|=\sqrt{(0-3)^2+(1-0)^2}=\sqrt{(-3)^2+(1)^2}=\left(\sqrt{9+1}\right)=\sqrt{10}$

$|RP|=\sqrt{(0+3)^2+(1-0)^2}=\sqrt{3^2+1^2}=\sqrt{10}$

And mid-point of P (−3, 0) and Q (3, 0) is

Where $x=\frac{-3+3}{2}=0$ and $y=\frac{0+0}{2}=0$

The point $(0, 1) \neq (0, 0)$

(iv) There is a unique midpoint of any two points in the plane.

Solved Exercise 9.3

1. Find the mid-point of the line segment joining each of the following pairs of points.

(a) A (9, 2), B (7, 2)

Solution: If R (x, y) is the mid point then:

$$R(x,y)=R\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=R\left(\frac{9+7}{2},\frac{2+2}{2}\right)=R\left(\frac{16}{2},\frac{4}{2}\right)=R(8,2)$$

(b) A (2,-6), B (3,-6)

Solution: If R (x, y) is the mid point then:

$$R(x,y)=R\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=R\left(\frac{2+3}{2},\frac{-6-6}{2}\right)$$

$$=R\left(\frac{5}{2},\frac{-12}{2}\right)=R(2.5,-6)$$

(c) A (-8, 1), B (6, 1)

Solution: If R (x, y) is the mid point then:

$$R(x,y)=R\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=R\left(\frac{-8+6}{2},\frac{1+1}{2}\right)$$

$$=R\left(\frac{-2}{2},\frac{2}{2}\right) \quad =R(-1,1)$$

(d) A (-4, 9), B (-4, -3)

Solution: If R (x, y) is the mid point then:

$$R(x,y)=R\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=R\left(\frac{-4-4}{2},\frac{9-3}{2}\right)$$

$$=R\left(-\frac{8}{2},\frac{6}{2}\right) \quad =R(-4,3)$$

(e) A (3, -11), B (3, -4)

Solution: If R (x, y) is the mid point then:

$$R(x,y)=R\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=R\left(\frac{3+3}{2},\frac{-11-4}{2}\right)$$

$$=R\left(\frac{6}{2},-\frac{15}{2}\right) =R(3,-7.5)$$

(f) A (0, 0), B (0, -5)

Solution: If R (x, y) is the mid point then:

$$R(x,y)=R\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=R\left(\frac{0+0}{2},\frac{0-5}{2}\right)$$

$$=R\left(\frac{0}{2},-\frac{5}{2}\right) =R(0,-2.5)$$

2. The end point P of a line segment PQ is (-3, 6) and its mid-point is (5, 8). Find the coordinates of the end point Q.

Solution: Since R (5, 8) is the mid point of P (–3, 6) and Q (x, y), then

$$5=\frac{x+(-3)}{2} \quad \text{and} \quad 8=\frac{y+6}{2}$$

$x-3=10$ $\qquad$ $y+6=16$

$x=10+3$ $\qquad$ $y=16-6$

$x=13$ $\qquad$ $y=10$

$\Rightarrow$ Q (x,y) = (13,10)

Hence the coordinates of the end point Q (x, y) = (13,10)

3. Prove that mid-point of the hypotenuse of a right triangle is equidistant from its three vertices P (-2, 5), Q (l, 3) and R(-l, 0).

Solution: Let P (–2, 5), Q(1, 3) and R (–l, 0) are the vertices of right triangle.

$$|PQ| = \sqrt{(1+2)^2+(3-5)^2}=\sqrt{(3)^2+(-2)^2}=\sqrt{9+4}=\sqrt{13}$$

$|QR| = \sqrt{(-1-1)^2+(0-3)^2} = \sqrt{(-2)^2+(-3)^2} = \sqrt{4+9} = \sqrt{13}$

$|PR| = \sqrt{(-1+2)^2+(0-5)^2} = \sqrt{(1)^2+(-5)^2} == \sqrt{1+25} = \sqrt{26}$

Since $|PQ|^2 + |QR|^2 = |PR|^2$

$(\sqrt{13})^2 + (\sqrt{13})^2 = (\sqrt{26})^2$

$13 + 13 = 26$

$26 = 26$

Hence PQR is a right triangle with PR as hypotenuse.

$$\text{Mid point of } |PR| = L\left(\frac{-2-1}{2}, \frac{5+0}{2}\right) = L\left(\frac{-3}{2}, \frac{5}{2}\right) = L(-1.5, 2.5)$$

Now $|LP| = \sqrt{[-2-(-1.5)]^2+(5-2.5)^2} = \sqrt{(-2+1.5)^2+(5-2.5)^2}$

$= \sqrt{(-0.5)^2+(2.5)^2} = \sqrt{0.25+6.25} = \sqrt{6.50} = 2.55$

$|LQ| = \sqrt{[1-(-1.5)]^2+(3-2.5)^2} = \sqrt{(1+1.5)^2+(0.5)^2}$

$= \sqrt{(2.5)^2+(0.5)^2} = \sqrt{6.25+0.25} = \sqrt{6.50} = 2.55$

$|LR| = \sqrt{[-1-(-1.5)]^2+(0-2.5)^2} = \sqrt{(-1+1.5)^2+(-2.5)^2}$

$= \sqrt{(0.5)^2+(-2.5)^2} = \sqrt{0.25+6.25} = \sqrt{6.50} = 2.55$

Since $|LP| = |LQ| = |LR| = 2.55$

Hence the mid point of the hypotenuse of a right triangle is equidistant from its three vertices.

4. If O (0, 0), A (3, 0) and B (3, 5) are three points in the plane, find M_1 and M_2 as mid-points of the line segments AB and OB respectively. Find $|M_1 M_2|$.

Solution: As O (0, 0), A (3, 0) and B (3, 5) are three points in the plane.

$$\text{Mid-point of AB} = M_1\left(\frac{3+3}{2}, \frac{0+5}{2}\right) = M_1\left(3, \frac{5}{2}\right) = M_1(3, 2.5)$$

$$\text{Mid-point of OB} = M_2\left(\frac{0+3}{2}, \frac{0+5}{2}\right) = M_2\left(\frac{3}{2}, \frac{5}{2}\right) = M_2(1.5,\ 2.5)$$

$$|M_1M_2| = \sqrt{(1.5-3)^2+(2.5-2.5)^2}$$

$$= \sqrt{(-1.5)^2+(0)^2}$$

$$= \sqrt{(-1.5)^2} = \sqrt{2.25} = 1.5 = \frac{3}{2}$$

5. Show that the diagonals of the parallelogram having vertices A(l, 2), B (4, 2), C(-l, -3) and D (-4, -3) bisect each other.
[Hint: The mid-points of the diagonals coincide]

Solution: Mid point of diagonal AC $= P\left(\frac{1-1}{2}, \frac{2-3}{2}\right)$

$$= P\left(\frac{0}{2}, \frac{-1}{2}\right) = P(0, -0.5)$$

Mid point of diagonal BD $= Q\left(\frac{4-4}{2}, \frac{2-3}{2}\right)$

$$= Q\left(\frac{0}{2}, -\frac{1}{2}\right) = Q(0, -0.5)$$

As mid point of diagonal AC and mid point of diagonal BC coincide. Hence they bisect each other.

6. The vertices of a triangle are P(4, 6), Q(-2, -4) and R(-8, 2). Show that the length of the line segment joining the mid-points of the line segments PR, QR is $\frac{1}{2}$ PQ.

Solution: $|PQ| = \sqrt{(-2-4)^2 + (-4-6)^2} = \sqrt{(-6)^2 + (-10)^2}$

$$= \sqrt{36+100} = \sqrt{136} = 2\sqrt{34}$$

Mid Point of QR $= M_1\left(\frac{-2-8}{2}, \frac{-4+2}{2}\right) = M_1\left(-\frac{10}{2}, -\frac{2}{2}\right) = M_1(-5, -1)$

Mid Point of PR $= M_2\left(\frac{4-8}{2}, \frac{6+2}{2}\right) = M_2\left(\frac{-4}{2}, \frac{8}{2}\right) = M_2(-2, 4)$

$$|M_1M_2| = \sqrt{[-2-(-5)]^2 + [4-(-1)]^2}$$

$$= \sqrt{(-2+5)^2 + (4+1)^2} = \sqrt{(3)^2 + (5)^2}$$

$$= \sqrt{9+25} = \sqrt{34} = \frac{1}{2}(2\sqrt{34}) = \frac{1}{2}|PQ|$$

Hence proved

Solved Review Exercise 9

1. Choose the correct answer.

(i) Distance between points (0, 0) and (1, 1) is
(a) 0 (b) 1 (c) 2 (d) $\sqrt{2}$

(ii) Distance between the points (1, 0) and (0, 1) is
(a) 0 (b) 1 (c) $\sqrt{2}$ (d) 2

(iii) Mid point of the points (2, 2) and (0, 0) is
(a) (1, 1) (b) (1, 0) (c) (0, 1) (d) (−1, −1)

(iv) Mid point of the points (2, -2) and (-2, 2) is
(a) (2, 2) (b) (−2, −2) (c) (0, 0) (d) (1, 1)

(v) A triangle having all sides equal is called:
(a) Isosceles (b) Scalene (c) Equilateral (d) None of these

(vi) A triangle having all sides different is called:
(a) Isosceles (b) Scalene (c) Equilateral (d) None of these

Solution: **(i)** d **(ii)** c **(iii)** a **(iv)** c **(v)** c **(vi)** b

2. Answer the following, which is true and which is false.

(i) A line has two end points. ________
(ii) A line segment has one end point. ________
(iii) A triangle is formed by three collinear points. ________
(iv) Each side of a triangle has two collinear vertices. ________
(v) The end points of each side of a rectangle are collinear. ________
(vi) All the points that lie on the x-axis are collinear. ________
(vii) Origin is the only point collinear with the points of both the axes separately. ________

Answers: **(i)** F **(ii)** F **(iii)** F **(iv)** T **(v)** T **(vi)** T **(vii)** T

3. Find the distance between the following pairs of points.

(i) (6, 3), (3, -3)
Solution:

$$\text{Let } A(6,3), B(3,-3), \text{ then}$$
$$|AB| = \sqrt{(3-6)^2+(-3-3)^2} = \sqrt{(-3)^2+(-6)^2}$$
$$= \sqrt{9+36} = \sqrt{45}$$

(ii) (7,5),(1,-1)
Solution:

$$\text{Let } A(7,5), B(1,-1), \text{ then}$$
$$|AB| = \sqrt{(1-7)^2+(-1-5)^2} = \sqrt{(-6)^2+(-6)^2}$$
$$= \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$$

(iii) (0,0), (-4,-3)
Solution:

$$\text{Let } A(0,0), B(-4,-3), \text{ then}$$
$$|AB| = \sqrt{(-4-0)^2+(-3-0)^2} = \sqrt{(-4)^2+(-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

4. Find the mid-point between following pairs of points.

(i) (6, 6), (4,-2)
Solution:

If $R(x,y)$ is the mid point, then

$$R(x,y) = R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = R\left(\frac{6+4}{2}, \frac{6-2}{2}\right)$$

$$= R\left(\frac{10}{2}, \frac{4}{2}\right) = R(5,2)$$

(ii) (-5, -7), (-7, -5)

Solution:

If $R(x,y)$ is the mid point then

$$R(x,y) = R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = R\left(\frac{-5-7}{2}, \frac{-7-5}{2}\right)$$

$$= R\left(-\frac{12}{2}, -\frac{12}{2}\right) = R(-6,-6)$$

(iii) (8, 0),(0, -12)

Solution:

If $R(x,y)$ is the mid point, then

$$R(x,y) = R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = R\left(\frac{8+0}{2}, \frac{0-12}{2}\right)$$

$$= R\left(\frac{8}{2}, -\frac{12}{2}\right) = R(4,-6)$$

5. Define the following

(i)	**Co-ordinate Geometry**	**(ii)**	**Collinear points**
(iii)	**Non-collinear points**	**(iv)**	**Equilateral Triangle**
(v)	**Scalene Triangle**	**(vi)**	**Isosceles Triangle**
(vii)	**Right Triangle**	**(viii)**	**Square**

Solution: (i) Co-ordinate Geometry:

Coordinate geometry is the study of geometrical shapes in the Cartesian plane (coordinate plane).

(ii) Collinear points:

Two or more than two points which lie on the same straight line are called collinear points.

(iii) Non-collinear points:

Two or more than two points which do not lie on the same straight line are called non-collinear points.

(iv) Equilateral Triangle:

If the lengths of all the three sides of a triangle are same, then the triangle is called an equilateral triangle.

(v) Scalene Triangle:

A triangle is called a scalene triangle if measures of all the three sides are different.

(vi) Isosceles Triangle:

An isosceles triangle is a triangle which has two of its sides with equal length while the third side has a different length.

(vii) Right Triangle:

A triangle in which one of the angles has measure equal to 90° is called a right angle triangle.

(viii) Square:

A square is a closed figure formed by four non-collinear points such that lengths of all sides are equal and measure of each angle is 90°.

SUMMARY

- If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points and d is the distance between them, then

$$d = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$

- The concept of non-Collinearity supports formation of the three-sided and four-sided shapes of the geometrical figures.
- The points P, Q and R are collinear if $|PQ| + |QR| = |PR|$
- The three points P, Q and R form a triangle if and only if they are non-collinear i.e., $|PQ| + |QR| > |PR|$
- If $|PQ| + |QR| < |PR|$ then no unique triangle can be formed by the points P, Q and R.
- Different forms of a triangle i.e., equilateral, isosceles, right angled and scalene are discussed in this unit.
- Similarly, the four-sided figures, square, rectangle and parallelogram are also discussed.

UNIT 10

CONGRUENT TRIANGLES

Unit Outlines

10.1 *Congruent Triangles*

STUDENTS LEARNING OUTCOMES

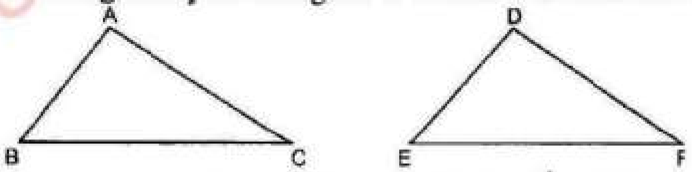
After studying this unit, the students will be able to:

- prove that in any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent.
- prove that if two angles of a triangle are congruent, then the sides opposite to them are also congruent.
- prove that in a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent.
- prove that if in the correspondence of two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent.

CONGRUENT TRIANGLES

Introduction:

Before proving the theorems, we will explain what is meant by 1-1 correspondence and congruency of triangles. We shall also state S.A.S. postulate.



Let there be two triangles ABC and DEF, the following six (1 – 1) correspondences can be established between ΔABC and ΔDEF.

In the correspondence $\Delta ABC \leftrightarrow \Delta DEF$ it means

$\angle A \leftrightarrow \angle D$	($\angle A$ corresponds to $\angle D$)
$\angle B \leftrightarrow \angle E$	($\angle B$ corresponds to $\angle E$)
$\angle C \leftrightarrow \angle F$	($\angle C$ corresponds to $\angle F$)
$\overline{AB} \leftrightarrow \overline{DE}$	($\overline{AB}$ corresponds to $\overline{DE}$)
$\overline{BC} \leftrightarrow \overline{EF}$	($\overline{BC}$ corresponds to $\overline{EF}$)

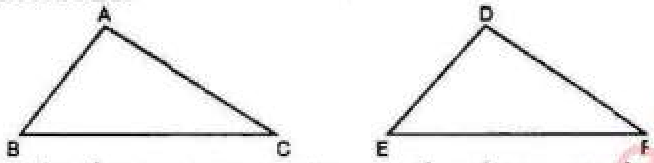
$\overline{CA} \leftrightarrow \overline{FD}$ ($\overline{CA}$ corresponds to $\overline{FD}$)

Congruency of Triangles:

Two triangles are said to be congruent written symbolically as, $\cong$, if there exists a correspondence between them such that all the corresponding sides and angles are congruent e.g.,

If $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, \overline{CA} \cong \overline{FD}$ and $\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$

Then $\Delta ABC \cong \Delta DEF$



Note: (i) These triangles are congruent w.r.t. the above mentioned choice of the (1–1) correspondence.

(ii) $\Delta ABC \cong \Delta ABC$ **(iii)** $\Delta ABC \cong \Delta DEF \leftrightarrow \Delta DEF \cong \Delta ABC$

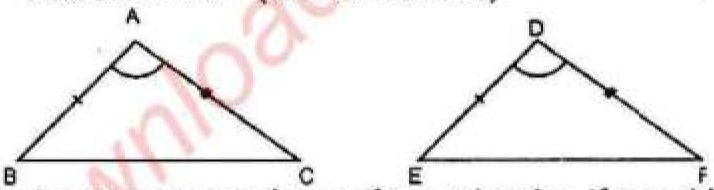
(iv) If $\Delta ABC \cong \Delta DEF$ and $\Delta ABC \cong \Delta PQR$, then $\Delta DEF \cong \Delta PQR$

In any correspondence of two triangles, if two sides and their included angle of one triangle are congruent to the corresponding two sides and their included angle of the other, then the triangles are congruent.

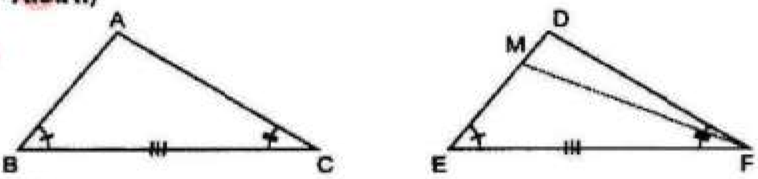
In the following figure, in $\Delta ABC \Leftrightarrow \Delta DEF$

If $\overline{AB} \cong \overline{DE}, \angle A \cong \angle D, \overline{AC} \cong \overline{DF}$

Then $\Delta ABC \cong \Delta DEP$ (S.A. S. Postulate)



Theorem 10.1.1: In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent.
(A.S.A = A.S.A.)



Solution: Given:

In $\Delta ABC \leftrightarrow \Delta DEF$, $\angle B \cong \angle E$, $\overline{BC} \cong \overline{EF}$, $\angle C \cong \angle F$

To Prove:

$\Delta ABC \cong \Delta DEF$

Construction:

Suppose $\overline{AB} \neq \overline{DE}$, take a point M on DE such that $\overline{AB} \cong \overline{ME}$. Join M to F

Proof:

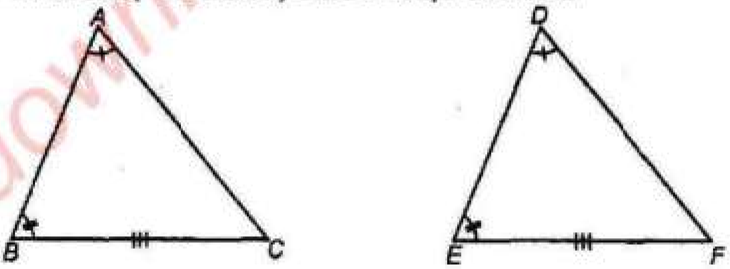
Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta MEF$	
$\overline{AB} \cong \overline{ME}$ ....(i)	Construction
$\overline{BC} \cong \overline{EF}$ ....(ii)	Given
$\angle B \cong \angle E$ ....(iii)	Given
$\therefore$ $\Delta ABC \cong \Delta MEF$	S.A.S. postulate
So, $\angle C \cong \angle MFE$	(Corresponding angles of congruent triangles)
But $\angle C \cong \angle DFE$	Given
$\therefore$ $\angle DFE \cong \angle MFE$	Both congruent to $\angle C$
This is possible only if D and M are the same points and $\overline{ME} \cong \overline{DE}$	$\because \overline{EM} \cong \overline{DE}$
So, $\overline{AB} \cong \overline{DE}$ ....(iv)	$\overline{AB} \cong \overline{ME}$ (construction) and $\overline{ME} \cong \overline{DE}$ (proved).
Thus from (ii), (iii) and (iv), we have $\Delta ABC \cong \Delta DEF$	S.A.S. postulate

Corollary:

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the correspondence side and angles of the other, then the triangles are congruent. (S.A.A. ≅ S.A.A.)

Solution: Given:

In $\Delta ABC \leftrightarrow \Delta DEF$, $\overline{BC} \cong \overline{EF}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$



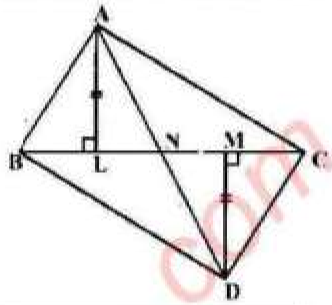
To Prove:

$\Delta ABC \cong \Delta DEF$

Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta DEF$	
$\angle B \cong \angle E$	Given
$\overline{BC} \cong \overline{EF}$	Given
$\angle C \cong \angle F$	$\angle A \cong \angle D$, $\angle B \cong \angle E$, (Given)
$\Delta ABC \cong \Delta DEF$	A.S.A. ≅ A.S.A.

Example: If ΔABC and ΔDCB are on the opposite sides of common base BC such that $\overline{AL} \perp \overline{BC}$, $\overline{DM} \perp \overline{BC}$ and $\overline{AL} \cong \overline{DM}$, then $\overline{BC}$ bisects $\overline{AD}$.



Solution: Given:

ΔABC and ΔDCB are on the opposite sides of $\overline{BC}$ such that $\overline{AL} \perp \overline{BC}$,
$\overline{DM} \perp \overline{BC}$, $\overline{AL} \cong \overline{DM}$, and $\overline{AD}$ is cut by $\overline{BC}$ at N.

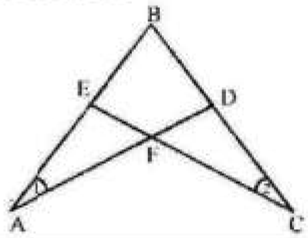
To Prove:
$\overline{AN} \cong \overline{DN}$

Proof:

Statements	Reasons
In $\Delta ALN \leftrightarrow \Delta DMN$	
$\overline{AL} \cong \overline{DM}$	Given
$\angle ALN \cong \angle DMN$	Each angle is right angle
$\angle ANL \cong \angle DNM$	Vertical angels
$\therefore$ $\Delta ALN \cong \Delta DMN$	S.A.A. $\cong$ S.A.A.
Hence $\overline{AN} \cong \overline{DN}$	Correspondence sides of $\cong$ Δs.

Solved Exercise 10.1

1. In the given figure, $\overline{AB} \cong \overline{CB}$, $\angle 1 \cong \angle 2$. Prove that $\Delta ABD \cong \Delta CBE$.



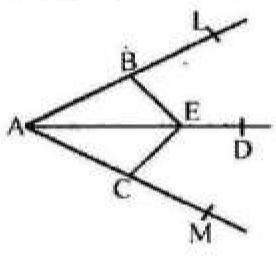
Solution:
Given:
$\overline{AB} \cong \overline{CB}$ and $\angle 1 \cong \angle 2$

To prove:
$\Delta\ ABD \cong \Delta\ CBE$

Proof:

Statements	Reasons
In $\Delta\ ABD \leftrightarrow \Delta\ CBE$	
$\overline{AB} \cong \overline{CB}$	Given
$\angle 1 \cong \angle 2$	Given
$\angle ADB \cong \angle CEB$	Right angles (90^o)
$\Delta ABD \cong \Delta\ CBE$	A.S.A postulates

2. From a point on the bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.



Solution: Given:

AD bisects of an angle $\angle BAC$ from point E, draw $\overline{EC} \perp \overline{AM}$ and $\overline{EB} \perp \overline{AL}$.

To prove:
$m\overline{EB} = m\overline{EC}$

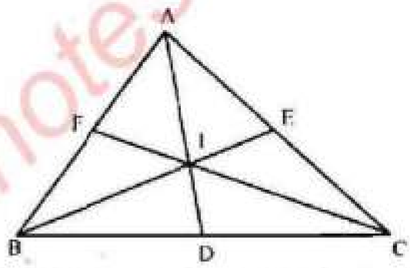
Construction:
$\angle BAC$ is bisected by $\overline{AD}$. Let E is a point on $\overline{AD}$.

Perpendicular to angle arm are drawn on $\overline{EB}$ and $\overline{EC}$.

Proof:

Statements	Reasons
In $\Delta AEB \leftrightarrow \Delta AEC$	
$m\angle BAE = m\angle CAE$	Bisection of $\angle BAC$
$m\angle EBA = m\angle ECA$	Both right angles
$m\overline{AE} = m\overline{AE}$	Common
$\therefore$ $\Delta AEB = \Delta AEC$	$A.S.A \cong A.S.A$
So $m\overline{BE} = m\overline{EC}$	Corresponding sides of congruent triangles

3. In a triangle ABC, the bisectors of $\angle B$ and $\angle C$ meet in a point I. Prove that I is equidistant from the three sides of ΔABC.



Solution: Given:

In ΔABC

$\overline{IF} \perp \overline{AB}$, $\overline{IE} \perp \overline{AC}$, $\overline{ID} \perp \overline{BC}$

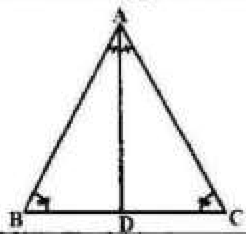
To prove: $\overline{ID} \cong \overline{IE} \cong \overline{IF}$

Construction: Join I with A, B and C.

Proof:

Statements	Reasons
In $\Delta BDI \leftrightarrow \Delta BFI$	
$\overline{BI} \cong \overline{BI}$	Common
$\angle BFI \cong \angle BDI$	Right angles (90°)
$\overline{BD} \cong \overline{BF}$	Perpendicular bisector of sides
$\Delta BDI \cong \Delta BFI$	S.A.S Postulates
Similarly, $\Delta IFA \cong \Delta IEA$	Corresponding sides of congruent triangles
and $\Delta IBD \cong \Delta ICD$	
So, $\overline{ID} \cong \overline{IE} \cong \overline{IF}$	

Theorem 10.1.2: If two angles of a triangle are congruent, then the sides opposite to them are also congruent.



Solution: Given: In ΔABC, $\angle B \cong \angle C$

To Prove: $\overline{AB} \cong \overline{AC}$

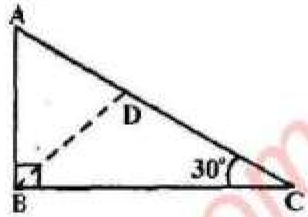
Construction:

Draw the bisector of $\angle A$, meeting $\overline{BC}$ at the point D.

Proof:

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta ACD$	
$\overline{AD} \cong \overline{AD}$	Common
$\angle B \cong \angle C$	Given
$\angle BAD \cong \angle CAD$	Construction
$\therefore$ $\Delta ABD \cong \Delta ACD$	$S.A.A. \cong S.A.A.$
Hence $\overline{AB} \cong \overline{AC}$	(Corresponding sides of congruent triangles)

Example-1: If one angle of a right angled triangle is of 30°, the hypotenuse is twice as long as the side opposite to the angle.



Solution: Given:

In ΔABC, $m\angle B = 90°$, $m\angle C = 30°$

To Prove:

$m\overline{AC} = 2m\overline{AB}$

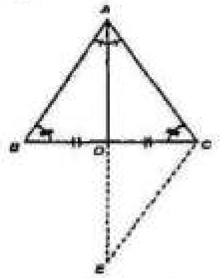
Construction:

At B, construct $\angle CBD$ of $30°$. Let $\overline{BD}$ cut $\overline{AC}$ at the point D.

Proof:

Statements	Reasons
In ΔABD, $m\angle A = 60°$	$m\angle ABC = 90°$, $m\angle C = 30°$
$m\angle ABD = m\angle ABC - m\angle CBD = 60°$	$m\angle ABC = 90°$, $m\angle CBD = 30°$
$\therefore m\angle ABD = 60°$	Sum of measures of $\angle$s of a Δ is $180°$
$\therefore \Delta ABD$ is equilateral	Each of its angles is equal to $60°$
$\overline{AB} \cong \overline{BD} \cong \overline{AD}$	Sides of equilateral Δ
In ΔBCD, $\overline{BD} \cong \overline{CD}$	$\angle C = \angle CBD$ (each of $30°$),
Thus $m\overline{AC} = m\overline{AD} + m\overline{CD}$	
$m\overline{AB} + m\overline{AB} = 2(m\overline{AB})$	$\overline{AD} \cong \overline{AB}$ and $\overline{CD} \cong \overline{BD} \cong \overline{AB}$

Example-2: If the bisector of an angle of a triangle bisects the side opposite to it, the triangle is isosceles.



Solution: Given:

In ΔABC, $\overline{AD}$ bisects $\angle A$ and $\overline{BD} \cong \overline{CD}$

To Prove:

$\overline{AB} \cong \overline{AC}$

Construction:

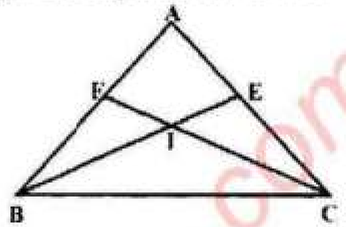
Produce $\overline{AD}$ to E, and take $\overline{ED} \cong \overline{AD}$. Joint C to E.

Proof:

Statements	Reasons
In $\Delta ADB \leftrightarrow \Delta EDC$	
$\overline{AD} \cong \overline{ED}$	Construction
$\angle ADB \cong \angle EDC$	Vertical angles
$\overline{BD} \cong \overline{CD}$	Given
$\therefore \Delta ADB \cong \Delta EDC$	S.A.S. Postulate
$\therefore \overline{AB} \cong \overline{CE}$ ………(i)	Correspondence sides of $\cong\Delta$s
$\angle BAD \cong \angle E$	Correspondence angles of $\cong\Delta$s
But $\angle BAD \cong \angle CAD$	Given
$\angle E \cong \angle CAD$	Each $\cong \angle BAD$
In ΔACE, $\overline{AC} \cong \overline{CE}$ ………(ii)	$\angle E \cong \angle CAD$ (proved)
Hence $\overline{AB} \cong \overline{AC}$	From (i) and (ii)

Solved Exercise 10.2

1. Prove that any two medians of an equilateral triangle are equal in measure.



Solution: Given:

ΔABC is an equilateral triangle.

$\therefore$ $\overline{AB} = \overline{AC} = \overline{BC}$

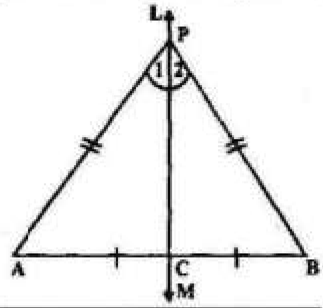
$\overline{BE}$ and $\overline{CF}$ are its medians.

To prove: $\overline{BE} \cong \overline{CF}$

Proof:

Statements	Reasons
In $\Delta BCE \leftrightarrow \Delta CBF$	
$\overline{BC} \cong \overline{BC}$	Common
$\angle FBC \cong \angle ECB$	Angles of equilateral triangles
$\overline{BF} \cong \overline{CE}$	Half of equal sides
$\therefore$ $\Delta BCE \cong \Delta CBF$	S.A.S. ≅ S.A.S.
Hence $\overline{CF} \cong \overline{BE}$	

2. Prove that a point, which is equidistant from the end points of a line segment, is on the right bisector of the line segment.



Solution: Given:

$\overline{AB}$ is a line segment. Point P is such that $\overline{PA} \cong \overline{PB}$

To prove:

Point P is on the right bisector of $\overline{AB}$.

Construction:

Join P to C, the mid point of $\overline{AB}$.

Proof:

Statements	Reasons
In $\Delta ACP \leftrightarrow \Delta BCP$	
$\overline{PA} \cong \overline{PB}$	Given
$\overline{PC} \cong \overline{PC}$	Common
$\overline{AC} \cong \overline{BC}$	Construction
$\Delta ACP \cong \Delta BCP$	S.S.S. ≅ S.S.S
$\angle ACP \cong \angle BCP$ ..... (i)	Corresponding angles of congruent triangles.
But $m\angle ACP + m\angle BCP = 180^\circ$ .... (ii)	Supplementary angles
$\therefore$ $m\angle ACP = m\angle BCP = 90^\circ$	From (i) and (ii)
Or $\overline{PC} \perp \overline{AB}$ ..... (iii)	$m\angle ACP = 90^\circ$ (Proved)
Also $\overline{CA} \cong \overline{CB}$ ..... (iv)	Construction
$\therefore$ $\overline{PC}$ is a right bisector of $\overline{AB}$ i.e., the point P is on the right bisector of $\overline{AB}$.	From (iii) and (iv).

Theorem 10.1.3: In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent (S.S.S ≅ S.S.S).

Solution: Given:

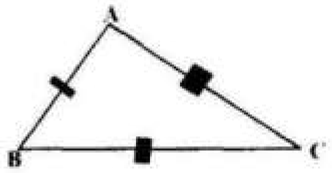
In $\Delta ABC \leftrightarrow \Delta DEF$

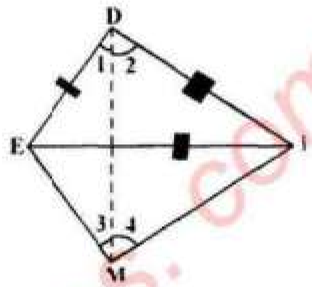
$\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$

and $\overline{CA} \cong \overline{FD}$

To Prove:

$\Delta ABC \cong \Delta DEF$





Construction:

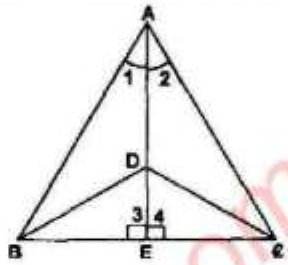
Suppose that in ΔDEF the side $\overline{EF}$ is not smaller than any of the remaining two sides. On $\overline{EF}$ construct a ΔMEF in which, $\angle FEM \cong \angle B$ and $\overline{ME} \cong \overline{AB}$. Join D and M. As shown in the above figures we label some of the angles as 1, 2, 3 and 4,

Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta MEF$	
$\overline{BC} \cong \overline{EF}$	Given
$\angle B \cong \angle FEM$	Construction
$\overline{AB} \cong \overline{ME}$	Construction
$\therefore$ $\Delta ABC \cong \Delta MEF$	S.A.S postulate
and $\overline{CA} \cong \overline{FM}$ ..... (i)	(corresponding sides of congruent triangles)
Also $\overline{CA} \cong \overline{FD}$ ..... (ii)	Given
$\therefore$ $\overline{FM} \cong \overline{FD}$	{From (i) and (ii)}
In ΔFDM	
$\angle 2 \cong \angle 4$ ..... (iii)	$\overline{FM} \cong \overline{FD}$ (proved)
Similarly $\angle 1 \cong \angle 3$ ..... (iv)	
$\therefore$ $m\angle 2 + m\angle 1 = m\angle 4 + m\angle 3$	{from (iii) and (iv)}
$\therefore$ $m\angle EDF = m\angle EMF$	
Now, in $\Delta DEF \leftrightarrow \Delta MEF$	
$\overline{FD} \cong \overline{FM}$	Proved
And $\angle EDF \cong \angle EMF$	Proved
$\overline{DE} \cong \overline{ME}$	Each one $\cong \overline{AB}$
$\therefore$ $\Delta DEF \cong \Delta MEF$	S.A.S. postulate
Also $\Delta ABC \cong \Delta MEF$	Proved
Hence $\Delta ABC \cong \Delta DEF$	Each $\Delta \cong \Delta MEF$ (proved)

Corollary:

If two isosceles triangles are formed on the same side of their common base, the line through their vertices would be the right bisector of their common base.



Solution: Given:

ΔABC and ΔDBC are formed on the same side of BC such that

$\overline{AB} \cong \overline{AC}, \overline{DB} \cong \overline{DC}$, $\overline{AD}$ meets $\overline{BC}$ at E.

To Prove:

$\overline{BE} \cong \overline{CE}, \overline{AE} \perp \overline{BC}$

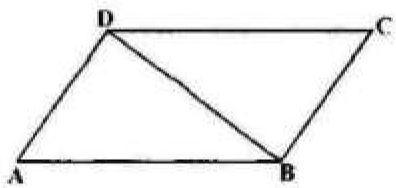
Proof:

Statements	Reasons
In $\Delta ADB \leftrightarrow \Delta ADC$	
$\overline{AB} \cong \overline{AC}$	Given
$\overline{DB} \cong \overline{DC}$	Given
$\overline{AD} \cong \overline{AD}$	Common
$\Delta ADB \cong \Delta ADC$	S.S.S $\cong$ S.S.S.
$\angle 1 \cong \angle 2$	Corresponding angles of $\cong$ Δs
In $\Delta ABE \leftrightarrow \Delta ACE$	
$\overline{AB} \cong \overline{AC}$	Given
$\angle 1 \cong \angle 2$	Proved
$\overline{AE} \cong \overline{AE}$	Common
$\therefore$ $\Delta ABE \cong \Delta ACE$	S.A.S. postulate
$\overline{BE} \cong \overline{CE}$	Corresponding sides of $\cong$ Δs
$\angle 3 \cong \angle 4$ .... (i)	Corresponding angles of $\cong$ Δs
$m\angle 3 + m\angle 4 = 180^\circ$ .... (ii)	Supplementary angles Postulate
$\therefore$ $m\angle 3 = m\angle 4 = 90^\circ$	From (i) and (ii)
Hence $\overline{AE} \perp \overline{BC}$	

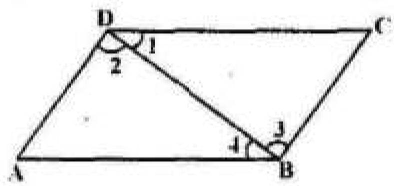
Corollary

An equilateral triangle is an equiangular triangle.

Solved Exercise 10.3



1. **In the figure,**
$\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$.
Prove that $\angle A \cong \angle C$,
$\angle ABC \cong \angle ADC$.



Solution: Given:

In quadrilateral ABCD,

$\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$

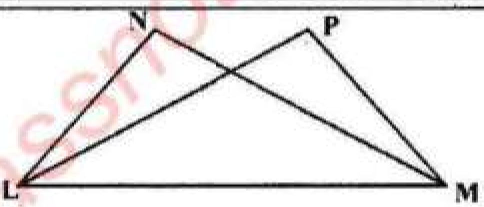
To prove:

$\angle A \cong \angle C$ and $\angle ABC \cong \angle ADC$.

Proof:

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta BCD$	
$\overline{AB} \cong \overline{DC}$	Given
$\overline{AD} \cong \overline{BC}$	Given
$\overline{BD} \cong \overline{BD}$	Common
$\Delta BCD \cong \Delta ABD$	S.S.S ≅ S.S.S
$\angle A \cong \angle C$	Due to congruency of triangles
$m\angle 1 = m\angle 4$ ...... (i)	
$m\angle 2 = m\angle 3$ ...... (ii)	
$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	Adding (i) and (ii)
$m\angle ADC = m\angle ABC$	
or $\angle ABC = \angle ADC$	

2. In the figure,
$\overline{LN} \cong \overline{MP}, \overline{MN} \cong \overline{LP}$
Prove that $\angle N \cong \angle P$,
$\angle NML \cong \angle PLM$.



Solution: Given:
$\overline{LN} \cong \overline{MP}$ and $\overline{LP} \cong \overline{MN}$

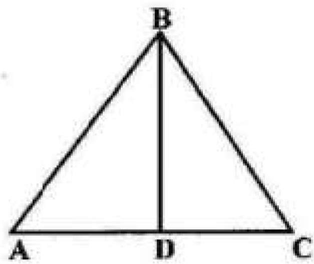
To Prove: $\angle N \cong \angle P$ and $\angle NML \cong \angle PLM$

Proof:

Statements	Reasons
In $\Delta LMP \leftrightarrow \Delta MLN$	
$\overline{LM} \cong \overline{LM}$	Common
$\overline{LP} \cong \overline{MN}$	Given
$\overline{LN} \cong \overline{MP}$	Given
$\Delta LMP \cong \Delta MLN$	S.S.S. ≅ S.S.S.
$\angle N \cong \angle P$	Congruent angles of congruent triangles
$\angle NML \cong \angle PLM$	Congruent angles of congruent triangles

3. Prove that the median bisecting the base of an isosceles triangle bisects the vertex angle and it is perpendicular to the base.

Solution: Given:
In ΔABC, $\overline{AB} \cong \overline{BC}$
BD bisects of base $\overline{AC}$
$\therefore$ $\overline{AD} \cong \overline{CD}$



To Prove: $\angle CBD \cong \angle ABD$ and $\overline{BD} \perp \overline{AC}$,
$\angle ABD \cong \angle CBD = 90^\circ$

Proof:

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta CBD$	
$\overline{AB} \cong \overline{BC}$	Given
$\overline{BD} \cong \overline{BD}$	Common
$\overline{AD} \cong \overline{CD}$	Given
$\Delta ABD \cong \Delta CBD$	S.S.S. ≅ S.S.S.
$\angle ABD \cong \angle CBD$	Corresponding angles of ≅ ΔS
$\angle ADB \cong \angle CDB$	Corresponding angles of ≅ ΔS
$m\angle ADB + m\angle CDB = 180°$	Supplementary angles
$2m\angle ADB = 180°$	$m\angle ADB \cong m\angle CDB$
$m\angle ADB = 90°$	Divide by 2
So, $\overline{BD} \perp \overline{AC}$	

Theorem 10.1.4: If in the correspondence of the two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent. (H.S≅H.S).

Solution: Given:

In $\Delta ABC \leftrightarrow \Delta DEF$

$\angle B \cong \angle E$ (right angles), $\overline{CA} \cong \overline{FD}$, $\overline{AB} \cong \overline{DE}$

To Prove: $\Delta ABC \cong \Delta DEF$

Construction:

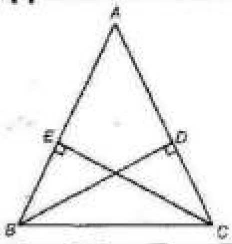
Produce $\overline{FE}$ to a point M such that $\overline{EM} \cong \overline{BC}$ and join the points D and M.

Proof:

Statements	Reasons
$m\angle DEF + m\angle DEM = 180°$ ...(i)	(Supplementary angles)
Now $m\angle DEF = 90°$ ....(ii)	Given
$\therefore$ $m\angle DEM = 90°$	From (i) and (ii)
In $\Delta ABC \leftrightarrow \Delta DEM$	
$\overline{BC} \cong \overline{EM}$	(construction)
$\angle ABC \cong \angle DEM$	(Each $\angle$ equal to 90°)
$\overline{AB} \cong \overline{DE}$	(Given)
$\therefore$ $\Delta ABC \cong \Delta DEM$	S.A.S. postulate
and $\angle C \cong \angle M$	(Corresponding angles of congruent triangles)
$\overline{CA} \cong \overline{MD}$	(Corresponding sides of congruent triangles)

Statements	Reasons
But $\overline{CA} \cong \overline{FD}$	(Given)
$\therefore$ $\overline{MD} \cong \overline{FD}$	Each is congruent to $\overline{CA}$
In ΔDMF	
$\angle F \cong \angle M$	$\overline{FD} \cong \overline{MD}$ (proved)
But $\angle C \cong \angle M$	(Proved)
$\angle C \cong \angle F$	(Each is congruent to $\angle M$)
In $\Delta ABC \leftrightarrow \Delta DEF$	
$\overline{AB} \cong \overline{DE}$	(Given)
$\angle ABC \cong \angle DEF$	(Given)
$\angle C \cong \angle F$	(Proved)
Hence $\Delta ABC \cong \Delta DEF$	(S.A.A. $\cong$ S.A.A)

Example: If perpendiculars from two vertices of a triangle to the opposite sides are congruent, then the triangle is isosceles.



Solution: Given:

In ΔABC, $\overline{BD} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$, such that $\overline{BD} \cong \overline{CE}$

To Prove:

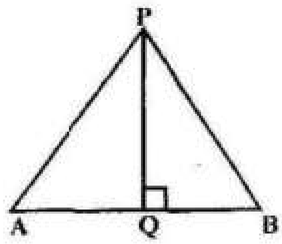
$\overline{AB} \cong \overline{AC}$

Proof:

Statements	Reasons
In $\Delta BCD \leftrightarrow \Delta CBE$	
$\angle BDC \cong \angle BEC$	$\overline{BD} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$ (given) $\Rightarrow$each angle $\Rightarrow 90°$
$\overline{BC} \cong \overline{BC}$	Common hypotenuse
$\overline{BD} \cong \overline{CE}$	Given
$\Delta BCD \cong \Delta CBE$	H.S. $\cong$ H.S.
$\angle BCD \cong \angle CBE$	Corresponding angles of $\cong$ Δs.
Thus $\angle BCA \cong \angle CBA$	
Hence $\overline{AB} \cong \overline{AC}$	In ΔABC, $\angle BCA \cong \angle CBA$

Solved Exercise 10.4

1. In ΔPAB of figure, $\overline{PQ} \perp \overline{AB}$ and $\overline{PA} \cong \overline{PB}$, prove that $\overline{AQ} \cong \overline{BQ}$ and $\angle APQ \cong \angle BPQ$.



Solution: Given:

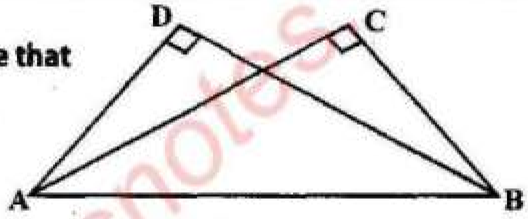
In ΔPAB, $\overline{PQ} \perp \overline{AB}$ and $\overline{PA} \cong \overline{PB}$

To Prove:

$\overline{AQ} \cong \overline{BQ}$ and $\angle APQ \cong \angle BPQ$

Statements	Reasons
In $\Delta PAQ \leftrightarrow \Delta PBQ$	
$\overline{PA} \cong \overline{PB}$	Given
$m\angle AQP \cong m\angle BQP$	Given
$\overline{PQ} \cong \overline{PQ}$	Common
$\Delta PAQ \cong \Delta PBQ$	H.S. ≅ H.S.
$\overline{AQ} \cong \overline{BQ}$	Corresponding sides of congruent triangles
$\angle APQ \cong \angle BPQ$	Corresponding angles of congruent triangles

2. In the figure,
$m\angle C = m\angle D = 90°$ and $\overline{BC} \cong \overline{AD}$. Prove that $\overline{AC} \cong \overline{BD}$, and $\angle BAC \cong \angle ABD$.

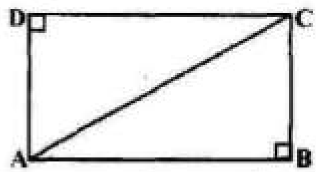


Solution: Given:
$m\angle C = m\angle D = 90°$ and $\overline{BC} \cong \overline{AD}$
To Prove: $\overline{AC} \cong \overline{BD}$ and $\angle BAC \cong \angle ABD$

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta ABD$	
$m\angle C = m\angle D = 90°$	Given
$\overline{BC} \cong \overline{AD}$	Given
$\overline{AB} \cong \overline{AB}$	Common
$\Delta ABC \cong \Delta ABD$	A.S.S. ≅ A.S.S
$\overline{AC} \cong \overline{BD}$	Congruent sides of congruent triangles
$\angle BAC \cong \angle ABD$	Congruent angles of congruent triangles

3. In the figure,
$m\angle B = m\angle D = 90°$ and $\overline{AD} \cong \overline{BC}$.
Prove that ABCD is a rectangle.

Solution: Given:
In rectangle ABCD
$m\angle B = m\angle D = 90°$ and $\overline{AD} \cong \overline{BC}$
To prove: ABCD is a rectangle.
Construction: Join A to C.



Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta ACD$	
$m\angle B = m\angle D = 90°$	Given
$\overline{AD} \cong \overline{BC}$	Given
$\overline{AC} \cong \overline{AC}$	Common
$\Delta ABC \cong \Delta ACD$	A.S.S. ≅ A.S.S
$m\angle A \cong m\angle C = 90°$	Right angle (90°)
Hence, ABCD is a rectangle.	

Solved Review Exercise 10

1. Which of the following statements are true and which are false?

(i) A ray has two end points. ______

(ii) In a triangle, there can be only one right angle. ______

(iii) Three points are said to be collinear, if they lie on same line. ______

(iv) Two parallel lines intersect at a point. ______

(v) Two lines can intersect only at one point. ______

(vi) A triangle of congruent sides has non-congruent angles. ______

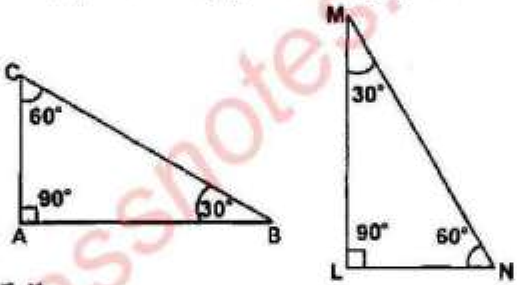
Answers: (i) F (ii) T (iii) T (iv) F (v) T (vi) F

2. If $\Delta ABC \cong \Delta LMN$, then

(i) $m\angle M \cong$ ———

(ii) $m\angle N \cong$ ———

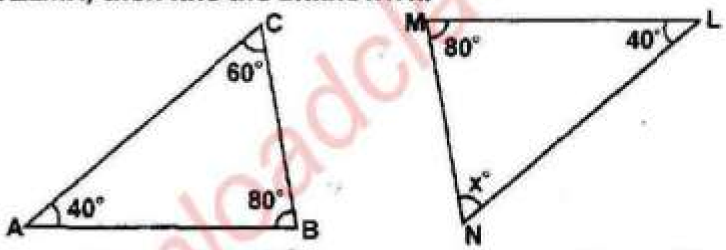
(iii) $m\angle A \cong$ ———



Solution:

(i) $m\angle B$ (ii) $m\angle C$ (iii) $m\angle L$

3. If $\Delta ABC \cong \Delta LMN$, then find the unknown x.



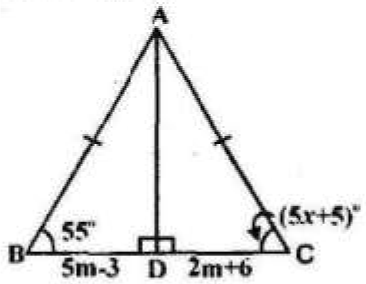
Solution: As, $m\angle A = m\angle L = 40^\circ$ and $m\angle B = m\angle N = 80^\circ$

So, $m\angle N = m\angle C$

$x = 60^\circ$

4. Find the value of unknowns for the given congruent triangles.

Solution: As triangles are congruent, so



$$m\angle C = m\angle B$$

$$\Rightarrow 5x^\circ + 5^\circ = 55^\circ$$

$$5x^\circ = 55^\circ - 5^\circ$$

$$5x^\circ = 50^\circ$$

$$x^\circ = 10^\circ$$

And $m\overline{BD} \cong m\overline{DC}$

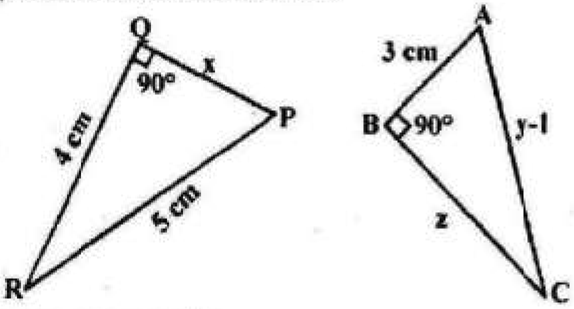
$$5m - 3 = 2m + 6$$

$$5m - 2m = 6 + 3$$

$$3m = 9$$

$$m = 3$$

5. If $\Delta PQR \cong \Delta ABC$, then find the unknowns.



Solution: As triangles are congruent, so

$$\overline{QP} \cong \overline{AB}$$
$$x = 3 \text{ cm}$$

And $\overline{BC} \cong \overline{QR}$

$$z = 4 \text{ cm}$$

And $\overline{AC} \cong \overline{PR}$

$$y - 1 = 5$$
$$y = 5 + 1$$
$$y = 6 \text{ cm}$$

SUMMARY

We stated and proved the following theorems:

- In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, the two triangles are congruent. (A.S.A ≅ A.S,A.)
- If two angles of a triangle are congruent, then the sides opposite to them are also congruent.
- In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent (S.S.S ≅ S.S.S).
- If in the correspondence of the two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent. (H.S ≅ H.S).
- Two triangles are said to the congruent, if there exists a correspondence between them such that all the corresponding sides and angles are congruent.

UNIT 11

PARALLELOGRAMS AND TRIANGLES

Unit Outlines	
11.1 *Parallelograms and*	11.2 *Triangles*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- prove that in a parallelogram
  - the opposite sides are congruent,
  - the opposite angles are congruent,
  - the diagonals bisect each other.
- prove that if two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.
- prove that the line segment, joining the midpoints of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- prove that the medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- prove that if three or more parallel lines make congruent segments on a transversal they also intercept congruent segments on any other line that cuts them

Introduction:

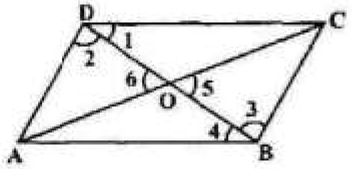
Before proceeding to prove the theorems in this unit the students are advised to recall definitions of polygons like parallelogram, rectangle, square, rhombus, trapezium etc. and in particular triangles and their congruency.

Theorem 11.1.1: In a parallelogram

(i) Opposite sides are congruent.

(ii) Opposite angles are congruent.

(iii) The diagonals bisect each other.



Solution: Given:

In a quadrilateral ABCD, $\overline{AB} \parallel \overline{DC}$, $\overline{BC} \parallel \overline{AD}$ and the diagonals $\overline{AC}$, $\overline{BD}$ meet each other at point O.

To Prove: (i) $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

(ii) $\angle ADC \cong \angle ABC$, $\angle BAD \cong \angle BCD$

(iii) $\overline{OA} \cong \overline{OC}$, $\overline{OB} \cong \overline{OD}$

Construction:

In the figure as shown, we label the angles as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$.

Proof:

Statements	Reasons
(i) In $\Delta ABD \leftrightarrow \Delta CDB$	
$\angle 4 \cong \angle 1$	Alternate angles
$\overline{BD} \cong \overline{BD}$	Common
$\angle 2 \cong \angle 3$	Alternate angles
$\therefore$ $\Delta ABD \cong \Delta CDB$	A.S.A. ≅ A.S.A.
So, $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	(Corresponding sides of congruent triangles)
and $\angle A \cong \angle C$	(Corresponding angles of congruent triangles)
(ii) Since	
$\angle 1 \cong \angle 4$ ...... (a)	Proved
And $\angle 2 \cong \angle 3$ ...... (b)	Proved
$\therefore$ $m\angle 1 + m\angle 2 = m\angle 4 + m\angle 3$	from (a) and (b)
Or $m\angle ADC = m\angle ABC$	
Or $\angle ADC \cong \angle ABC$	
And $\angle BAD \cong \angle BCD$	Proved in (i)
(iii) In $\Delta BOC \leftrightarrow \Delta DOA$	
$\overline{BC} \cong \overline{AD}$	Proved in (i)
$\angle 5 \cong \angle 6$	vertical angles
$\angle 3 \cong \angle 2$	Proved
$\therefore$ $\Delta BOC \cong \Delta DOA$	(A.A.S. ≅ A.A.S.)
Hence $\overline{OC} \cong \overline{OA}$, $\overline{OB} \cong \overline{OD}$	(Corresponding sides of congruent triangles)

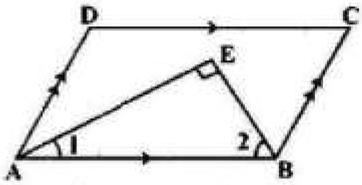
Corollary:

Each diagonal of a parallelogram bisects it into two congruent triangles.

Example: The bisectors of two angles on the same side of a parallelogram cut each other at right angles.

Solution: Given:

A parallelogram ABCD, in which $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$. The bisectors of $\angle A$ and $\angle B$ cut each other at E.



To Prove: $m\angle E = 90°$

Construction: Name the angles $\angle 1$ and $\angle 2$ as shown in the figure.

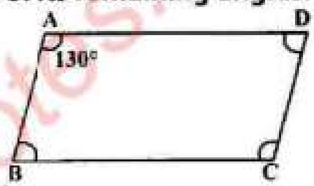
Proof:

Statements	Reasons
$m\angle 1 + m\angle 2$ $= \frac{1}{2}(m\angle BAD + m\angle ABC)$	$\begin{cases} m\angle 1 = \frac{1}{2} m\angle BAD, \\ m\angle 2 = \frac{1}{2} m\angle ABC \end{cases}$

Statements	Reasons
$= \frac{1}{2}(80^\circ)$ $= 90^\circ$	Int. angles on the same side of $\overline{AB}$ which cuts ‖ segments $\overline{AD}$ and $\overline{BC}$ are supplementary.
Hence in ΔABE, $m\angle E = 90^\circ$	$m\angle 1 + m\angle 2 = 90^\circ$ (proved)

Solved Exercise 11.1

1. One angle of a parallelogram is 130°. Find the measures of its remaining angles.



Solution: Given:

In parallelogram ABCD, $m\angle A = 130^\circ$

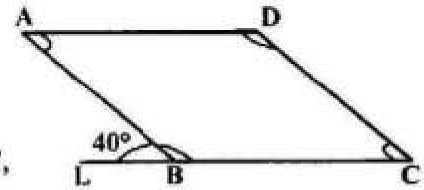
To prove:

$m\angle B = ?$, $m\angle C = ?$, $m\angle D = ?$,

Proof:

Statements	Reasons
$\angle A = 130^\circ$	Given
$\angle A \cong \angle C$	Opposite angles are equal
So, $\angle C = 130^\circ$	
$m\angle A + m\angle B = 180^\circ$	Angle formed with same side of parallelogram
$130^\circ + m\angle B = 180^\circ$	
$m\angle B = 180^\circ - 130^\circ$	
$m\angle B = 50^\circ$	
As $\angle B = \angle D$	Opposite angles are equal.
So $\angle D = 50^\circ$	

2. One exterior angle formed on producing one side of a parallelogram is 40°, Find the measures of its interior angles.



Solution: Given:

In parallelogram ABCD, $m\angle ABL = 40^\circ$

To prove:

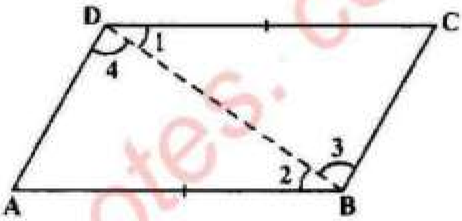
$m\angle A = ?$, $m\angle B = ?$, $m\angle C = ?$, $m\angle D = ?$,

Proof:

Statements	Reasons
$m\angle ABC + m\angle ABL = 180^\circ$	Supplementary angles
$m\angle ABC + 40^\circ = 180^\circ$	
$m\angle ABC = 180^\circ - 40^\circ$	
$m\angle B = 140^\circ$	
As $m\angle B \cong m\angle D$	Opposite angles are equal
So, $m\angle D = 140^\circ$	
$m\angle A + m\angle B = 180^\circ$	

Statements	Reasons
$m\angle A + 140^\circ = 180^\circ$	Opposite angles are equal
$m\angle A = 180^\circ - 140^\circ$	
$m\angle A = 40^\circ$	Opposite angles are equal.
As $m\angle C = m\angle A$	
So $m\angle C = 40^\circ$	

Theorem 11.1.2: If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.



Solution: Given:

In a quadrilateral ABCD, $\overline{AB} \cong \overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

To Prove:

ABCD is a parallelogram.

Construction:

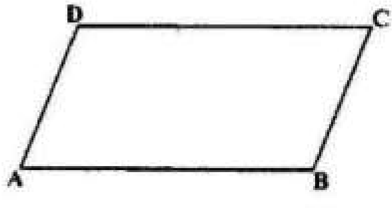
Join the point B to D and in the figure, name the angles as indicated: $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$

Proof:

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta CDB$	
$\overline{AB} \cong \overline{DC}$	Given
$\angle 2 \cong \angle 1$	Alternate angles
$\overline{BD} \cong \overline{BD}$	Common
$\therefore \Delta ABD \cong \Delta CDB$	S.A.S. postulate
Now $\angle 4 \cong \angle 3$ .....(i)	(Corresponding angles of congruent triangles)
$\therefore \overline{AD} \parallel \overline{BC}$ .....(ii)	From (i)
and $\overline{AD} = \overline{BC}$ .....(iii)	Corresponding sides of congruent Δs
Also $\overline{AB} \parallel \overline{DC}$ .....(iv)	Given
Hence ABCD is a parallelogram	From (ii) – (iv)

Solved Exercise 11.2

1. Prove that a quadrilateral is a parallelogram if its
   (a) opposite angles are congruent.
   (b) diagonals bisect each other.



(a) Solution: Given:

In a quadrilateral ABCD, $\angle A \cong \angle C$ and $\angle B \cong \angle D$

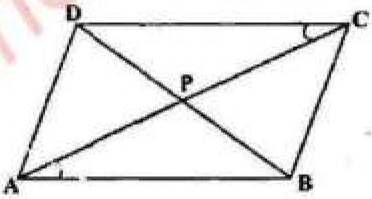
To Prove:

Quadrilateral ABCD is a parallelogram.

Proof:

Statements	Reasons
$m\angle A = m\angle C$ ...... (i)	Given
$m\angle B = m\angle D$ ...... (ii)	Proved
$m\angle A + m\angle B = m\angle C + m\angle D$ ...... (iii)	Common
$m\angle A + m\angle B + m\angle C + m\angle D = 360^\circ$	(Corresponding angles of congruent triangles.)
$m\angle A + m\angle B + m\angle A + m\angle B = 360^\circ$	
$2(m\angle A + m\angle B) = 360^\circ$	
$m\angle A + m\angle B = 180^\circ$	
So, $\overline{BC} \parallel \overline{AD}$	
$m\angle B + m\angle C = 180^\circ$	
So, $\overline{AB} \parallel \overline{DC}$	
Hence, quadrilateral ABCD is a parallelogram.	

(b) Solution:



Given: The diagonal AC and BD intersect each other at point P. So, $\overline{PA} \cong \overline{PC}$, $\overline{PB} \cong \overline{PD}$.

To Prove:

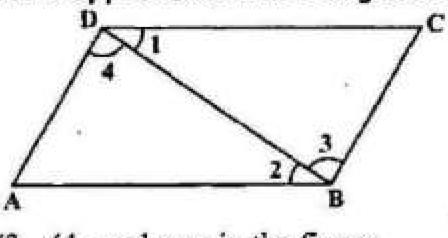
Quadrilateral ABC is a parallelogram.

Proof:

Statements	Reasons
In $\Delta PAB \leftrightarrow \Delta PCD$	
$\overline{PA} \cong \overline{PC}$	Given
$\angle APB \cong \angle CPD$	Vertical angles
$\overline{PB} \cong \overline{PD}$	Given
$\Delta PAB \cong \Delta PCD$	S.A.S. Postulate
$\overline{AB} \cong \overline{DC}$	(Corresponding sides of congruent triangles.)
$\angle PAB = \angle PCD$	(Corresponding angles of congruent triangles.)
$\overline{AB} \parallel \overline{DC}$	
Hence, quadrilateral ABCD is a parallelogram.	

2. Prove that a quadrilateral is a parallelogram if its opposite sides are congruent.

Solution:



Given: In a quadrilateral ABCD. $\overline{AB} \cong \overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

To Prove:

ABCD is a parallelogram.

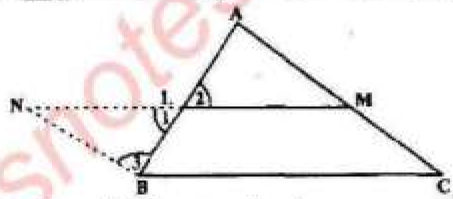
Construction:

Join B to D and name the angles $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, as shown in the figure.

Proof:

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta CDB$	
$\overline{AB} \cong \overline{DC}$	Given
$\angle 2 \cong \angle 1$	Vertical angles
$\overline{BD} \cong \overline{BD}$	Common
$\therefore$ $\Delta ABD \cong \Delta CDB$	S.A.S. Postulate
And $\angle 4 \cong \angle 3$ ..... (i)	(Corresponding angles of congruent triangles.)
$\therefore$ $\overline{AD} \parallel \overline{BC}$ ...... (ii)	From (i)
And $\overline{AB} \parallel \overline{DC}$ ...... (iii)	Given
Hence, ABCD is a parallelogram.	From (ii) and (iii)

Theorem 11.1.3: The line segment, joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.



Solution:

Given: In ΔABC, the mid-points of $\overline{AB}$ and $\overline{AC}$ are L and M respectively.

To Prove: $\overline{LM} \parallel \overline{BC}$ and $m\overline{LM} = \frac{1}{2} m\overline{BC}$

Construction: Join M to L and produce $\overline{ML}$ to N such that $\overline{LM} \cong \overline{LN}$. Join N to B and in the figure, name the angles as $\angle 1$, $\angle 2$ and $\angle 3$ as shown.

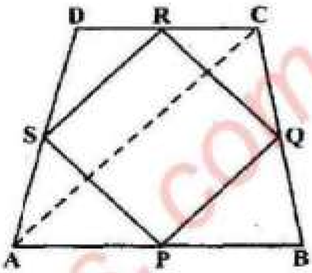
Proof:

Statements	Reasons
In $\Delta BLN \leftrightarrow \Delta ALM$	
$\overline{BL} \cong \overline{AL}$	Given
$\angle 1 \cong \angle 2$	Vertical angles
$\overline{NL} \cong \overline{ML}$	Construction
$\therefore$ $\Delta BLN \cong \Delta ALM$	S.A. S. postulate
$\therefore$ $\angle A \cong \angle 3$ ...... (i)	(Corresponding angles of congruent triangles)
and $\overline{NB} \cong \overline{AM}$ ......(ii)	(Corresponding sides of congruent triangles)
But $\overline{NB} \parallel \overline{AM}$	From (i), alternate $\angle$s
Thus $\overline{NB} \parallel \overline{MC}$ ......(iii)	(M is mid-point of $\overline{AC}$)
$\overline{MC} \cong \overline{AM}$ ......(iv)	Given
$\overline{NB} \cong \overline{MC}$ ......(v)	{From (ii) and (iv)}
$\therefore$ BCMN is a parallelogram	from (iii) and (v)
$\overline{BC} \parallel \overline{LM}$ or $\overline{BC} \parallel \overline{NL}$	(Opposite sides of a parallelogram BCMN)
$\overline{BC} \cong \overline{NM}$ ...... (vi)	(Opposite sides of a parallelogram)
$m\overline{LM} = \frac{1}{2} m\overline{NM}$ ....(vii)	Construction
Hence $m\overline{LM} = \frac{1}{2} m\overline{BC}$	{from (vi) and (vii)}

Note: that instead of producing $\overline{ML}$ to N, we can take N on $\overline{LM}$ produced.

Example: The line segments, joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram.

Solution: Given:



A quadrilateral ABCD, in which P is the mid-point of $\overline{AB}$, Q is the mid-point of $\overline{BC}$, R is the mid-point of $\overline{CD}$, S is the mid-point of $\overline{DA}$. P is joined to Q, Q is joined to R, R is joined to S and S is joined to P.

To Prove:

PQRS is a parallelogram.

Construction: Join A to C.

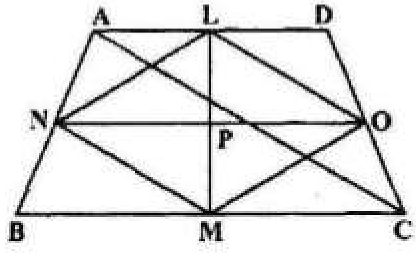
Proof:

Statements	Reasons
In ΔDAC,	
$\overline{SR} \parallel \overline{AC}$	S is the mid-point of $\overline{DA}$
$m\overline{SR} = \frac{1}{2} m\overline{AC}$	R is the mid-point of $\overline{CD}$
In ΔBAC,	
$\overline{PQ} \parallel \overline{AC}$	P is the mid-point of $\overline{AB}$
$m\overline{PQ} = \frac{1}{2} m\overline{AC}$	Q is the mid-point of $\overline{BC}$
$\overline{SR} \parallel \overline{PQ}$	Each $\parallel$ $\overline{AC}$
$m\overline{SR} = m\overline{PQ}$	Each $= \frac{1}{2} m\overline{AC}$
Thus PQRS is a parallelogram	$\overline{SR} \parallel \overline{PQ}$, $m\overline{SR} = m\overline{PQ}$ (proved)

Solved Exercise 11.3

1. Prove that the line-segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.

Solution:



Given:

A quadrilateral ABCD, the midpoints of the opposite sides of which are joined together.

To prove:

$\overline{NP} \cong \overline{PO}$ and $\overline{LP} \cong \overline{PM}$

Construction:

Join L, M, N and O in order.

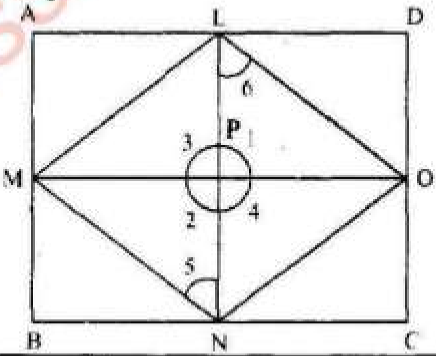
Proof:

Statements	Reasons
In Δ ABC	
$\overline{NM} \parallel \overline{AC}$	N is the mid Point of $\overline{AB}$
$m\,\overline{NM} = m\,\frac{1}{2}\,\overline{AC}$	M is the mid point of $\overline{BC}$
In Δ ADC	
$\overline{LO} \parallel \overline{AC}$	L is the mid point of $\overline{AD}$
$m\overline{LO} = m\,\frac{1}{2}\,\overline{AC}$	O is the mid point of $\overline{DC}$
$\overline{NM} \parallel \overline{LO}$	Each $\parallel \overline{AC}$
$m\overline{NM} = m\,\overline{LO}$	Each $= \frac{1}{2}\,m\overline{AC}$
$\overline{NP} \cong \overline{PO}$	$\overline{NP} \parallel \overline{PO}$, $m\overline{NP} = m\overline{PO}$
$\overline{LP} \cong \overline{PM}$	Diagonals of parallelogram

2. Prove that the line-segments joining the mid-points of the opposite sides of a rectangle are the right-bisectors of each other.
[Hint: Diagonals of a rectangle are congruent.]

Solution: Given:

ABCD is rectangle having L, M, N and O midpoints.



To prove:

$\overline{MO}$ and $\overline{LN}$ intersect each other at right angle.

Construction:

Join L to M and O and N.

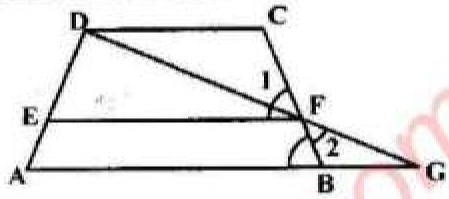
Proof:

Statements	Reasons
In $\Delta NPM \leftrightarrow \Delta LPO$	
$\angle 1 \cong \angle 2$	Vertical angles
$\angle 5 \cong \angle 6$	Corresponding angles
$\overline{MP} \cong \overline{LO}$	Diagonals of rectangles are congruent
$\Delta NPM \cong \Delta LPO$	A.A.S. postulate
$\overline{MP} \cong \overline{PO}$ .... (i)	(corresponding sides of congruent triangles)
$\overline{LP} \cong \overline{PN}$ ..... (ii)	
$\angle 1 \cong \angle 2$	
$\angle 3 \cong \angle 4$	Complementary postulate
$m\angle 1 + m\angle 3 = 180^o$	
$m\angle 2 + m\angle 3 = 180^o$	
$m\angle 1 + m\angle 2 = 180^o$	
$m\angle 1 = m\angle 2 = 90^o$	Supplementary postulate

3. Prove that the line-segment passing through the mid-point of one side and parallel to another side of a triangle also bisects the third side.

Solution: Given:

Trapezium ABCD having mid-points, E, F on $\overline{AD}$ and $\overline{BC}$ respectively.



To prove:

$\overline{AB} \parallel \overline{DC} \parallel \overline{EF}$

Construction:

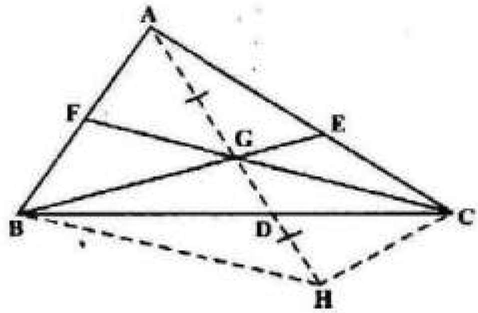
Join F and D take G intersecting point of $\overline{AB}$ and $\overline{DF}$.

Proof:

Statements	Reasons
In $\Delta DCF \leftrightarrow \Delta GBF$	
$\overline{BF} \cong \overline{FC}$	Mid points on $\overline{BC}$
$\angle 1 \cong \angle 2$	Vertical angles
$\angle DCF \cong \angle FBG$	Corresponding angles
$\Delta DCF \cong \Delta GBF$	S.A.A. Postulates.
$\overline{AF} \cong \overline{FG}$	
In ΔABG	
$\overline{EF} \parallel \overline{BG}$	
$\overline{EF} \parallel \overline{AD} \parallel \overline{BC}$	

Theorem 11.1.4: The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

Solution: Given: ΔABC



To Prove: The medians of the ΔABC are concurrent and the point of concurrency is the point of trisection of each median.

Construction:

Draw two medians $\overline{BE}$ and $\overline{CF}$ of the ΔABC which intersect each other at point G. Join A to G and produce it to point H such that $\overline{AG} \cong \overline{GH}$. Join H to the points B and C.

$\overline{AH}$ intersects $\overline{BC}$ at the point D.

Proof:

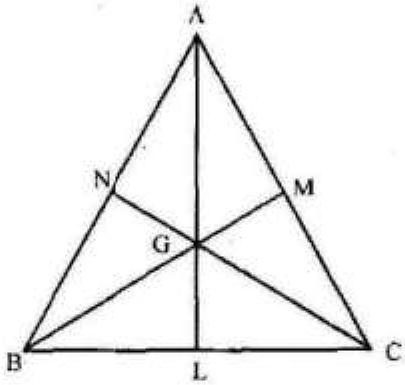
Statements	Reasons
In ΔACH,	
$\overline{GE} \parallel \overline{HC}$	G and E are mid-points of sides $\overline{AH}$ and $\overline{AC}$ respectively
or $\overline{BE} \parallel \overline{HC}$ ....(i)	G is a point of $\overline{BE}$
Similarly $\overline{CF} \parallel \overline{HB}$ ....(ii)	
$\therefore$ BHCG is a parallelogram	From (i) and (ii)

Statements	Reasons
and $m\overline{GD} = \frac{1}{2} m\overline{GH}$ ….. (iii)	(Diagonals BC and GH of a parallelogram BHCG intersect each other at point D)
$\overline{BD} \cong \overline{CD}$ $\overline{AD}$ is a median of ΔABC	
Medians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ pass through the point G	(G is the intersecting point of $\overline{BE}$ and $\overline{CF}$ and $\overline{AD}$ pass through it)
Now $\overline{GH} \cong \overline{AG}$ …… (iv)	Construction
$\therefore$ $m\overline{GD} = \frac{1}{2} m\overline{AG}$	From (iii) and (iv)
And G is the point of trisection of $\overline{AD}$ …….(v)	
Similarly it can be proved that G is also the point of trisection of $\overline{CF}$ and $\overline{BE}$	

Solved Exercise 11.4

1. The distances of the point of concurrency of the medians of a triangle from its vertices are respectively 1.2 cm, 1.4 cm and 1.6 cm. Find the lengths of its medians.

Solution:



$\overline{LG} = \frac{1}{2}\ \overline{AG} = \frac{1}{2}\ (1.2) = 0.6$ cm

$\overline{AL} = \overline{AG} + \overline{LG} = 1.2 + 0.6 = 1.8$ cm

$\overline{MG} = \frac{1}{2}\ (1.4) = 0.7$ cm

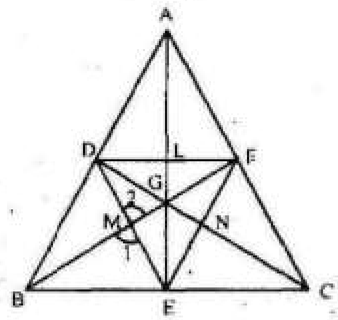
$\overline{BM} = 1.4 + 0.7 = 2.1$ cm

$\overline{NG} = \frac{1}{2}\ \overline{CG} = \frac{1}{2}\ (1.5) = 0.75$ cm

$\overline{CN} = \overline{CG} + \overline{NG} = 1.5 + 0.75$

$= 2.25$ cm

2. Prove that the point of concurrency of the medians of a triangle and the triangle which is made by joining the mid-points of its sides is the same.



Solution:

Given:

The mid points of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are D, E and F respectively.

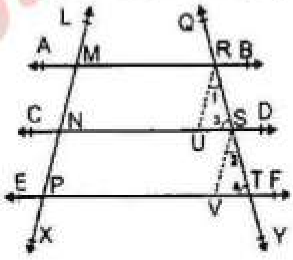
To prove:

Centroid of ABC and DEF is same.

Proof:

Statements	Reasons
$\overline{DF} \parallel \overline{BE}$	
$\overline{BE} \cong \overline{EC}$	Mid points of $\overline{BC}$
$\overline{BE} \cong \overline{DF}$	Construction
$\overline{DB} \parallel \overline{FE}$	Construction
$\overline{DB} \cong \overline{FE}$	
In $\Delta BME \leftrightarrow \Delta DMF$	
$\overline{BE} \cong \overline{DF}$	Proved
$\angle 1 \cong \angle 2$	Vertical angles
$\angle DEB \cong \angle FDE$	Corresponding angles
$\Delta BMF = \Delta BME$	S.A.A Postulates
$\overline{DM} \cong \overline{ME}$	(Corresponding sides of congruent triangles)

Theorem 11.1.5: If three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.



Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$

The transversal $\overleftrightarrow{LX}$ intersects $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ at the points M, N and P respectively, such that $\overline{MN} \cong \overline{NP}$. The transversal $\overleftrightarrow{QY}$ intersects them at points R, S and T respectively.

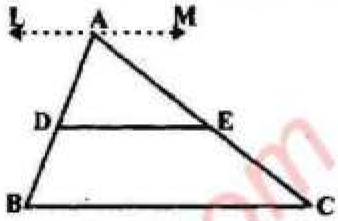
To Prove: $\overline{RS} \cong \overline{ST}$

Construction: From R, draw $\overline{RU} \parallel \overleftrightarrow{LX}$, which meets $\overleftrightarrow{CD}$ at U. From S, draw $\overline{SV} \parallel \overleftrightarrow{LX}$ which meets $\overleftrightarrow{EF}$ at V. As shown in the figure let the angles be labelled as $\angle 1$, 2, $\angle 3$ and $\angle 4$.

Proof:

Statements	Reasons
MNUR is a parallelogram	$\overline{RU} \parallel \overline{LX}$ (construction) $\overline{AB} \parallel \overline{CD}$ (given)
$\therefore$ $\overline{MN} \cong \overline{RU}$ ....(i)	(Opposite sides of a parallelogram)
Similarly, $\overline{NP} \cong \overline{SV}$ .... (ii)	
But $\overline{MN} \cong \overline{NP}$ .... (iii)	Given
$\therefore$ $\overline{RU} \cong \overline{SV}$	{From (i), (ii) and (iii)}
Also $\overline{RU} \parallel \overline{SV}$	Each is $\parallel$ $\overline{LX}$ (construction.)
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding angles
and $\angle 3 \cong \angle 4$	Corresponding angles
In $\Delta RUS \leftrightarrow \Delta SVT$	
$\overline{RU} \cong \overline{SV}$	Proved
$\angle 1 \cong \angle 2$	Proved
$\angle 3 \cong \angle 4$	Proved
$\Delta RUS \cong \Delta SVT$	S.A.A. $\cong$ S.A.A.
Hence $\overline{RS} \cong \overline{ST}$	(Corresponding sides of congruent triangles)

COROLLARIES: A line, through the mid-point of one side, parallel to another side of a triangle, bisects the third side.



Given: In ΔABC, D is the mid-point of $\overline{AB}$. $\overline{DE} \parallel \overline{BC}$ which cuts $\overline{AC}$ at E.

To Prove: $\overline{AE} \cong \overline{EC}$

Construction: Through A. draw $\overleftrightarrow{LM} \parallel \overline{BC}$.

Proof:

Statements	Reasons
Intercepts cut by $\overleftrightarrow{LM}$, $\overleftrightarrow{DM}$, $\overleftrightarrow{BC}$ on $\overline{AC}$ are congruent, i.e., $\overline{AE} \cong \overline{EC}$.	Intercepts cut by parallels $\overleftrightarrow{LM}, \overleftrightarrow{DE}$, $\overleftrightarrow{BC}$ on $\overline{AB}$ are congruent (given)

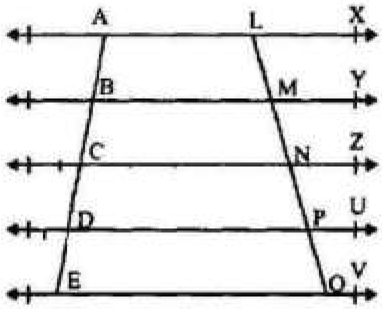
Corollaries:

(1) The parallel line from the mid-point of one non-parallel side of a trapezium to the parallel sides bisects the other non-parallel side.

(2) If one side of a triangle is divided into congruent segments, the line drawn from the point of division parallel to the other side will make congruent segments on third side.

Solved Exercise 11.5

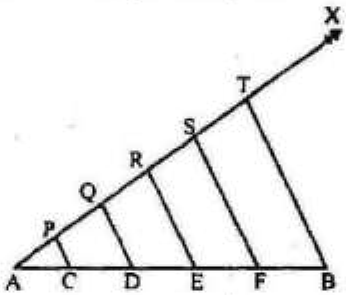
1. In the given figure,
$\overleftrightarrow{AX} \parallel \overleftrightarrow{BY} \parallel \overleftrightarrow{CZ} \parallel \overleftrightarrow{DU} \parallel \overleftrightarrow{EV}$ and
$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE}$.
If $m\overline{MN} = 1$ cm, then find the length of $\overline{LN}$ and $\overline{LQ}$.



Solution:

LM = MN = NP = PQ = 1 cm;
LN = LM + MN = 1 + 1 = 2cm
LQ = LM + MN + NP + PQ = 1+1+1+1 = 4 cm

2. Take a line segment of length 5.5 cm and divide it into five congruent parts.
[Hint: Draw an acute angle $\angle BAX$. On $\overrightarrow{AX}$ take $\overline{AP} \cong \overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong \overline{ST}$.
Join T to B. Draw lines parallel to $\overline{TB}$ from the points P, Q, R and S.]



Solution: Construction:

(i) Draw a line segment $\overline{AB}$ of length 5.5 cm.
(ii) Draw an acute angle $\angle$ BAX.
(iii) Draw arcs on $\overrightarrow{AX}$ of proper radius which intersects at point P, Q, R, S and T.
(iv) Join T and B.
(v) With the help of set square, draw lines $\overline{PC}$, $\overline{QD}$, $\overline{RE}$ and $\overline{SF}$ parallel to $\overline{TB}$.
(vi) The point C, D, E and F divide $\overline{AB}$ into 5 equivalent parts each of length 1.1cm.

Solved Review Exercise 11

1. Fill in the blanks.

(i) In a parallelogram opposite sides are ————.

(ii) In a parallelogram opposite angles are————.

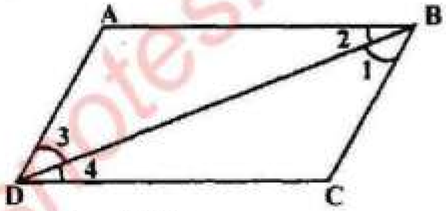
(iii) Diagonals of a parallelogram————each other at a point.

(iv) Medians of a triangle are ————,

(v) Diagonal of a parallelogram divides the parallelogram into two ____triangles.

Solution: (i) parallel (ii) equal (iii) intersect
(iv) concurrent (v) congruent

2. In parallelogram ABCD



(i) $m\overline{AB}$ _____ $m\overline{DC}$

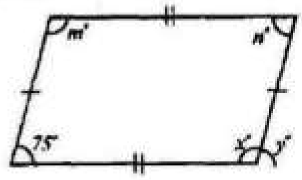
(ii) $m\overline{BC}$ _____ $m\overline{AD}$

(iii) $m\angle 1 \cong$ _____

(iv) $m\angle 2 \cong$ _____

Solution: (i) $\cong$ (ii) $\cong$ (iii) $m\angle 3$ (iv) $m\angle 4$

3. Find the unknowns in the given figure.



Solution:

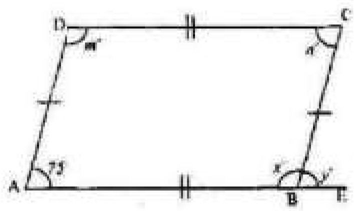
As $\angle C = \angle A \quad \Rightarrow n^\circ = 75^\circ$

$m^\circ + 75^\circ = 180^\circ \Rightarrow m^\circ = 180^\circ - 75^\circ = 105^\circ$

And $\angle B = \angle D$

$x^\circ = m^\circ$

$x^\circ = 105^\circ$



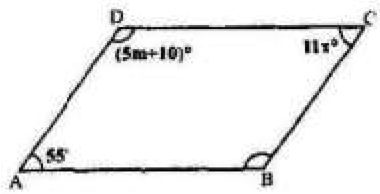
And $x^\circ + y^\circ = 180^\circ$

$y = 180^\circ - x^\circ$

$y = 180^\circ - m^\circ$

$y = 180^\circ - 105^\circ = 75^\circ$

4. If the given figure ABCD is a parallelogram, then find x, m.



Solution: As $\angle C = \angle A$

$11x^\circ = 55^\circ$

$x^\circ = 5^\circ$

So $\angle C = 11x^\circ = 11(5) = 55^\circ$

Now $\angle D = \angle B$

$(5m + 10)^\circ = 125^\circ$

$5m + 10 = 125^\circ$

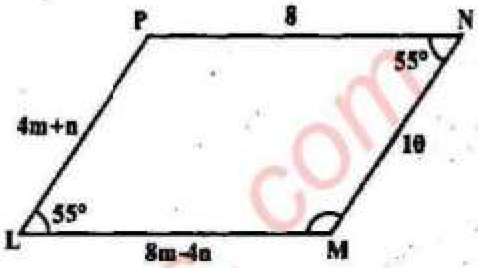
$5m = 125^\circ - 10^\circ$

$5m = 115^\circ$

$m = 23^\circ$

So $\angle D = 5m + 10 = 5(23) + 10 = 115 + 10 = 125^\circ$

5. The given figure LMNP is a parallelogram. Find the value of m, n.



Solution: $\Rightarrow$ $4m + n = 10$ .... (i)

$8m - 4n = 8$ ..... (ii)

By solving eq. (i) and eq. (ii), we get

$6n = 12 \quad \Rightarrow \quad n = 2$

Put $n = 2$ in eq. (i), we get

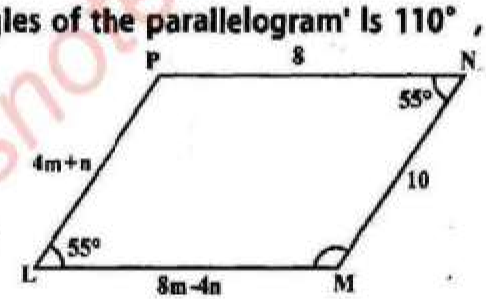
$4m + 2 = 10$

$4m = 10 - 2$

$4m = 8$

$m = 2$

6. In the question 5, sum of the opposite angles of the parallelogram' Is 110° , find the remaining angles.



Solution: We know that

$\angle L + \angle M + \angle N + \angle P = 360^\circ$

$\angle 55^\circ + \angle M + \angle 55^\circ + \angle P = 360^\circ$

As $\angle L = \angle N = 55^\circ$ and $\angle M = \angle P$

$\therefore \quad 2\angle M + 110^\circ = 360^\circ$

$2\angle M = 360^\circ - 110 = 250^\circ$

$\Rightarrow \quad \angle M = 125^\circ = \angle P$

SUMMARY

In this unit we discussed the following theorems and used them to solve some exercises. They are supplemented by unsolved exercises to enhance applicative skills of the students.

- In a parallelogram
  - (i) Opposite sides are congruent.
  - (ii) Opposite angles are congruent.
  - (iii) The diagonals bisect each other.
- If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.
- The line segment, joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- If three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.

UNIT 12

LINE BISECTORS AND ANGLE BISECTORS

Unit Outlines

12.1 *Bisector of a Line Segment and* 12.2 *Bisector of an Angle*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- prove that any point on the right bisector of a line segment is equidistant from its end points.
- prove that any point equidistant from the end points of a line segment is on the right bisector of it.
- prove that the right bisectors of the sides of a triangle are concurrent.
- prove that any point on the bisector of an angle is equidistant from its arms.
- prove that any point inside an angle, equidistant from its arms, is on the bisector of it.
- prove that the bisectors of the angles of a triangle are concurrent.

Introduction

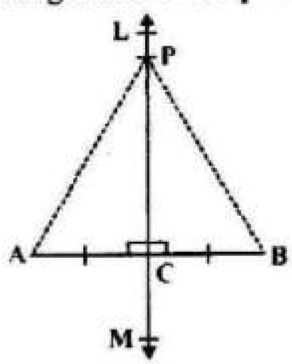
In this unit we will prove theorems and their converses, if any, about right bisector of a line segment and bisector of an angle. But before that it will be useful to recall the following definitions.

Right Bisector of a Line Segment:

A line is called a right bisector of a line segment if it is perpendicular to the line segment and passes through its mid point.

Bisector of an Angle:

A ray is called a bisector of line segment if it divides the angle into two equal parts. A ray BP is called the bisector of $\angle ABC$ if P is a point in the interior of the angle and $\angle ABP = \angle PBC$.



Theorem 12.1.1: Any point on the right bisector of a line segment is equidistant from its end points.

Solution: Given:

A line LM intersects the line segment AB at the point C such that $\overline{LM} \perp \overline{AB}$ and $\overline{AC} \cong \overline{BC}$. P is a point on $\overline{LM}$.

To Prove: $\overline{PA} \cong \overline{PB}$

Construction:

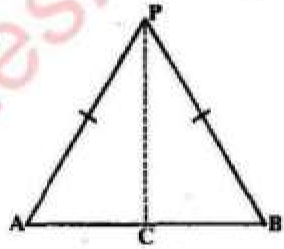
Take a point P on LM. Join P to the points A and B.

Proof:

Statements	Reasons
In $\Delta ACP \leftrightarrow \Delta BCP$	
$\overline{AC} \cong \overline{BC}$	Given
$\angle ACP \cong \angle BCP$	Given $\overline{PC} \perp \overline{AB}$, so that each $\angle$at C = 90°
$\overline{PC} \cong \overline{PC}$	common
$\therefore$ $\Delta ACP \cong \Delta BCP$	S.A.S. postulate
Hence $\overline{PA} \cong \overline{PB}$	Corresponding sides of congruent triangles

Theorem 12.1.2: {Converse of Theorem 12.1.1}

Any point equidistant from the end points of a line segment is on the right bisector of it.



Solution: Given:

$\overline{AB}$ is a line segment. Point P is such that $\overline{PA} \cong \overline{PB}$.

To Prove: The point P is on the right bisector of $\overline{AB}$.

Construction:

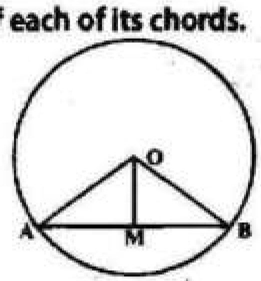
Joint P to C, the midpoint of $\overline{AB}$.

Proof:

Statements	Reasons
In $\Delta ACP \longleftrightarrow \Delta BCP$	
$\overline{PA} \cong \overline{PB}$	Given
$\overline{PC} \cong \overline{PC}$	Common
$\overline{AC} \cong \overline{BC}$	Construction
$\therefore$ $\Delta ACP \cong \Delta BCP$	S.S.S. $\cong$ S.S.S.
$\angle ACP \cong \angle BCP$ ...... (i)	(Corresponding angles of congruent triangles)
But $m\angle ACP + m\angle BCP = 180°$ ..... (ii)	Supplementary angles
$\therefore$ $m\angle ACP = m\angle BCP = 90°$	From (i) and (ii)
i.e., $\overline{PC} \perp \overline{AB}$ .....(iii)	$m\angle ACP = 90°$ (proved)
Also $\overline{CA} \cong \overline{CB}$ .....(iv)	Construction
$\therefore$ $\overline{PC}$ is a right bisector of $\overline{AB}$	From (ii) and (iv)
i.e., the point P is on the right bisector of $\overline{AB}$.	

Solved Exercise 12.1

1. Prove that the centre of a circle is on the right bisectors of each of its chords.



Solution: Given: A circle of radius 'r' with centre at 'O'. $\overline{AB}$ is the chord of circle.

To Prove: Point 'O' lies on the perpendicular bisector of chord $\overline{AB}$.

Construction: Take point M as the mid point of $\overline{AB}$. Join O with M, A and B.

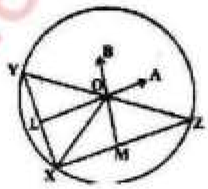
Proof:

Statements	Reasons
In $\Delta AMO \leftrightarrow \Delta BMO$	
$\overline{AO} \cong \overline{BO}$	Radius of same circle
$\overline{AM} \cong \overline{BM}$	Construction
$\overline{OM} \cong \overline{OM}$	Common
$\Delta AMO \cong \Delta BMO$	S.S.S ≅ S.S.S
$m\angle AMO \cong m\angle BMO$ ... (i)	Corresponding angle of congruent triangle
$m\angle AMO + m\angle BMO = 180°$ ...(ii)	Supplementary angles
So, $m\angle AMO \cong m\angle BMO = 90°$	From (i) and (ii)

By this it is proved that the centre of a circle is on the right bisector of its chord.

2. Where will be the centre of a circle passing through three non-collinear points? And why?

Sol: Given: X,Y,Z are three non-collinear points. A circle is passing through these points.



To Prove: To find out the centre of the circle.

Construction: Draw $\perp$ bisectors $\overline{LA}$ and $\overline{MB}$ of line segments $\overline{XY}$ and $\overline{XZ}$ respectively which intersect each other at point "O". Join "O" with X,Y and Z.

Proof:

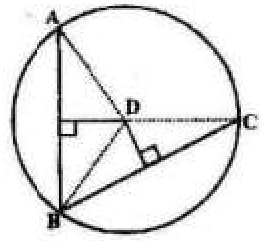
Statements	Reasons
$m\overline{OX} \cong m\overline{OZ}$ .... (i)	Point "O" is on the right bisector of $\overline{XZ}$.
$m\overline{OX} \cong m\overline{OY}$ .... (ii)	Point "O" is on the right bisector of $\overline{XY}$.
$m\overline{OX} \cong m\overline{OY} \cong m\overline{OZ}$	From (i) & (ii)

Since, Point "O" is a common point of points X,Y,Z therefore, point "O" is the center of the circle and points X,Y and Z are at equidistant from "O".

3. Three villages P, Q and R are not on the same line. The people of these villages want to make a Children Park at such a place which is equidistant from these three villages. After fixing the place of Children Park, prove that the Park is equidistant from the three villages.

Solution: Given:

Three points A, B, C are not on a line.



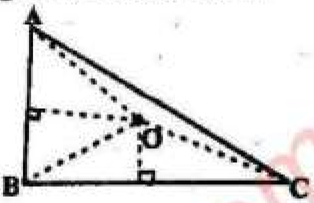
To prove: $m\overline{AD} = m\overline{BD} = m\overline{CD}$

Construction: Draw perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$. Both of them intersect each other at D. With center at D Draw a circle passing through A, B and C.

Proof:

Statements	Reasons
If D is the place fixed for children park.	
Then, $m\overline{AD} = m\overline{BD}$	Radius of Same Circle
$m\overline{BD} = m\overline{CD}$	Radius of Same Circle
$m\overline{AD} = m\overline{BD} = m\overline{CD}$	Radius of Same Circle
Hence, D is equidistant from point A, B & C.	

Theorem 12.1.3: The right bisectors of the sides of a triangle are concurrent.



Solution: Given: ΔABC

To Prove: The right bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ are concurrent.

Construction: Draw the right bisectors of $\overline{AB}$ and $\overline{BC}$ which meet each other at the point O. Join O to A, B and C.

Proof:

Statements	Reasons
$\overline{OA} \cong \overline{OB}$ ...... (i)	(Each point on right bisector of a segment is equidistant from its end points)
$\overline{OB} \cong \overline{OC}$ ..... (ii)	as in (i)
$\overline{OA} \cong \overline{OC}$ ..... (iii)	from (i) and (ii)
$\therefore$ Point O is on the right bisector of $\overline{CA}$. .....(iv)	O is equidistant from A and C
But Point O is On the right bisector of $\overline{AB}$ and of $\overline{BC}$ ... (v)	Construction
Hence the right bisectors of the three sides of a triangle are concurrent at O.	From (iv) and (v)

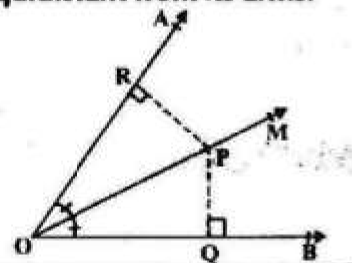
Observe that:

(a) The right bisectors of the sides of an acute triangle intersect each other inside the triangle.

(b) The right bisectors of the sides of a right triangle intersect each other on the hypotenuse.

(c) The right bisectors of the sides of an obtuse triangle intersect each other outside the triangle.

Theorem 12.1.4: Any point on the bisector of an angle is equidistant from its arms.



Solution: Given:

A point P is on $\overrightarrow{OM}$, the bisector of $\angle AOB$.

To Prove: $\overline{PQ} = \overline{PR}$

i.e., P is equidistant from $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Construction:

Draw $\overline{PR} \perp \overrightarrow{OA}$ and $\overline{PQ} \perp \overrightarrow{OB}$

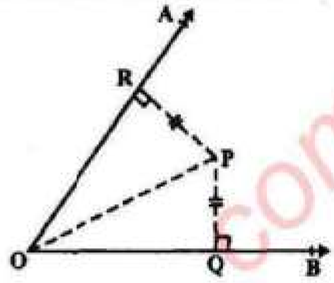
Proof:

Statements		Reasons
In	$\Delta POQ \leftrightarrow \Delta POR$	
	$\overline{OP} \cong \overline{OP}$	Common
	$\angle PQO \cong \angle PRO$	Construction
	$\angle POQ \cong \angle POR$	Given
$\therefore$	$\Delta POQ \cong \Delta POR$	S.A.A. $\cong$ S.A.A,
Hence	$\overline{PQ} \cong \overline{PR}$	(Corresponding sides of congruent triangles)

Theorem 12.1.5: (Converse of Theorem 12.1.4)

Any point inside an angle, equidistant from its arms, is on the bisector of it.

Solution: Given:



Any point P lies inside $\angle AOB$ such that $\overline{PQ} \cong \overline{PR}$, where $\overline{PQ} \perp \overline{OB}$ and $\overline{PR} \perp \overline{OA}$.

To Prove: Point P is on the bisector of $\angle AOB$.

Construction:

Join P to O.

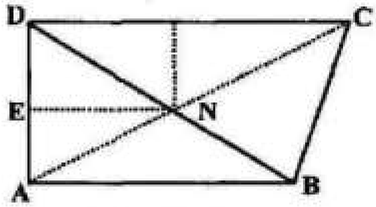
Proof:

Statements	Reasons
In $\Delta POQ \leftrightarrow \Delta POR$	
$\angle PQO \cong \angle PRO$	Given (right angles)
$\overline{PO} \cong \overline{PO}$	Common
$\overline{PQ} \cong \overline{PR}$	Given
$\therefore$ $\Delta POQ \cong \Delta POR$	H.S. $\cong$ H.S.
Hence, $\angle POQ \cong \angle POR$	Corresponding angels of congruent triangles
i.e., P is on the bisector of $\angle AOB$.	

Solved Exercise 12.2

1. **In a quadrilateral ABCD, $\overline{AB} \cong \overline{BC}$ and the right bisectors of $\overline{AD}$, $\overline{CD}$ meet each other at point N. Prove that $\overline{BN}$ is a bisector of $\angle ABC$.**

Solution: Given:



A quadrilateral ABCD, $\overline{AB} \cong \overline{BC}$. A point N such that $\angle NED = 90°$

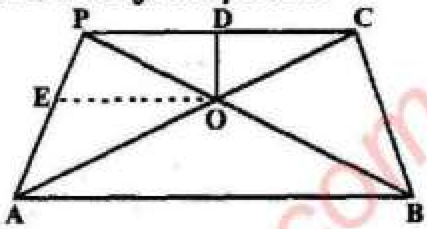
To Prove:

$\angle ABN = \angle CBN$

Proof:

Statements	Reasons
$\overline{AN} \cong \overline{DN}$ .....(i)	Given
$\overline{DN} \cong \overline{CN}$ .....(ii)	Given
$\overline{AN} \cong \overline{CN}$ .....(ii)	From eq. (i) and (ii)
In $\Delta ABN \leftrightarrow \Delta BCN$	
$\overline{BN} \cong \overline{BN}$	Common
$\overline{AB} \cong \overline{BC}$	Given
$\overline{AN} \cong \overline{CN}$	Proved
$\Delta ABN \cong \Delta BCN$	S.S.S Postulate
$\angle ABN \cong \angle CBN$	Corresponding angles of Congruent triangles

2. The bisectors of $\angle A$, $\angle B$ and $\angle C$ of a quadrilateral ABCP meet each other at point O. Prove that the bisector of $\angle P$ will also pass through the point O.



Solution: Given: A quadrilateral ABCP and bisector of $\angle A$, $\angle B$ and $\angle C$ passes through O

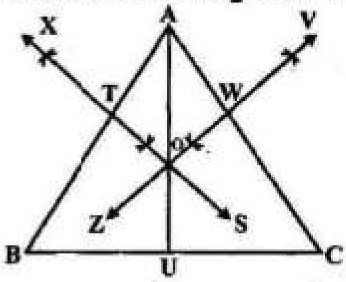
To Prove: Bisector of $\angle P$ passes through 'O'

Construction: Draw two perpendiculars of PC and AP from 'O' meet at D and E respectively. Draw the bisector of $\angle P$.

Proof:

Statements	Reasons
In $\Delta EOP \leftrightarrow \Delta DOP$	
$\overline{OP} \cong \overline{OP}$	Common
$\angle PEO \cong \angle PDO$	Construction
$\angle EPO \cong \angle DPO$	Construction
$\Delta OEP \cong \Delta ODE$	S.A.S. $\cong$ S.A.S
So, $\overline{EO} \cong \overline{OD}$	Corresponding sides of Congruent triangles.
O is on bisector of $\angle P$.	Any point on the bisector of an angle is equidistant from its arms.

3. Prove that the right bisectors of congruent sides of an isosceles triangle and its altitude are concurrent.



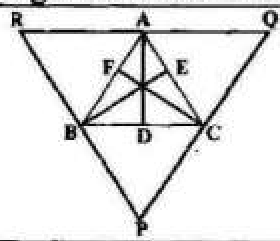
Solution: Given: ABC is an isosceles triangle in which $\overline{AB} \cong \overline{AC}$. T and W are mid points of sides $\overline{AB}$ and $\overline{AC}$. $\overline{XS}$ and $\overline{ZV}$ are the right bisectors of two sides whereas $\overline{AU}$ is the altitude of triangle.

To Prove: The right bisector of sides and altitude intersect each other at same point.

Proof:

Statements	Reasons
$\overline{XS}$ and $\overline{ZV}$ intersect at point 'O'	The bisector of two sides.
$\overline{AU} \perp \overline{BC}$	Altitude of isosceles triangle.
But $\overline{AB} \cong \overline{AC}$	Given
$\therefore$ Point A is on the bisector of $\overline{BC}$.	Point A is equidistance from B and C.
$\therefore$ $\overline{AU}$ is the perpendicular bisector of $\overline{BC}$.	$\overline{AU} \perp BC$
$\therefore$ $\overline{AU}$ also passes through O.	The bisectors of a triangle are concurrent.

4. Prove that the altitudes of a triangle are concurrent.



Solution: Given:

In ΔABC, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are altitudes.

To Prove: The altitudes of triangle are concurrent.

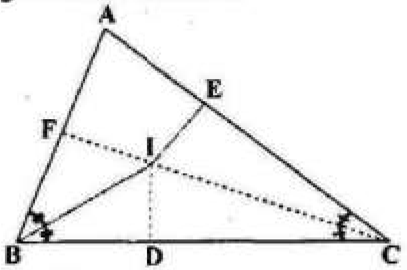
Concurrent:

Draw $\overline{RP} \parallel \overline{AC}$, $\overline{RQ} \parallel \overline{BC}$ and $\overline{PQ} \parallel \overline{AB}$.

Proof:

Statements	Reasons
ABCQ is a parallelogram	$\overline{QC} \parallel \overline{AB}$, $\overline{AQ} \parallel \overline{BC}$
$\overline{AQ} \parallel \overline{BC}$ ....(i)	Opposite sides of a parallelogram
$\overline{AR} \parallel \overline{BC}$ ....(ii)	Opposite sides of a parallelogram
$\therefore$ $\overline{AQ} \cong \overline{AR}$	From (i) and (ii)
$\therefore$ A is the mid point of $\overline{RQ}$	
$\overline{RQ} \parallel \overline{BC}$	Construction
$\therefore$ $\overline{AD} \perp \overline{RQ}$	As $\overline{AD} \perp \overline{BC}$
Therefore $\overline{AD}$ is the perpendicular bisector of $\overline{RQ}$	$\therefore$ A is the mid point of $\overline{RQ}$.
Similarly, it can be proved that $\overline{BE}$ is bisector of $\overline{RP}$ and $\overline{CF}$ is the bisector of $\overline{PQ}$.	
$\therefore$ $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ intersect each other.	$\therefore$ The perpendicular bisector of ΔPQR bisect each other

Theorem 12.1.6: The bisectors of the angles of a triangle are concurrent.



Solution: Given:

ΔABC

To Prove:

The bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Construction:

Draw the bisectors of $\angle B$ and $\angle C$ which intersect at point I. From I, draw $\overline{IF} \perp \overline{AB}$, $\overline{IE} \perp \overline{CA}$ and $\overline{ID} \perp \overline{BC}$.

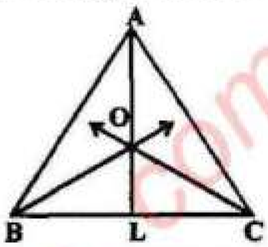
Proof:

Statements	Reasons
$\overline{ID} \cong \overline{IF}$	(Any point on bisector of an angle is equidistant from its arms)
Similarly,	
$\overline{ID} \cong \overline{IE}$	
$\therefore$ $\overline{IE} \cong \overline{IF}$	Each = ID, proved.
So, the point I is on the bisector of $\angle A$ ......(i)	
Also the point I is on the bisectors of $\angle ABC$ and $\angle BCA$ ...... (ii)	
Thus the bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent at I.	Construction {from (i) and (ii)}

Note: In practical geometry, by constructing angle bisectors of a triangle, we shall verify that they are concurrent.

Solve Exercise 12.3

1. Prove that the bisectors of the angles of base of an isosceles triangle intersect each other on its altitude.



Solution: Given:

Δ ABC and bisector of two external angles ABC and ACB intersect each other in point 'O'.

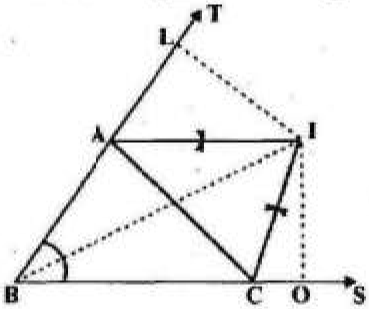
To Prove:

The altitude of triangle ABC passes through 'O'

Proof:

Statements	Reasons
In $\Delta AOB \leftrightarrow \Delta AOC$	
$\overline{AO} \cong \overline{OA}$	Common
$\angle OBA \cong \angle OCA$	Opposite angles of congruent sides
$\overline{AB} \cong \overline{AC}$	Given
Hence, $\Delta AOB \cong \Delta AOC$	S.A.S ≅ S.A.S
So, $\overline{OB} \cong \overline{OC}$	Corresponding sides of congruent triangles
So, it is proved that the altitude of triangle ABC passes through O.	

2. Prove that the bisectors of two exterior and third interior angle of a triangle are concurrent.



Solution: Given:

ΔABC is a triangle which produces side $\overline{AB}$ to point T and $\overline{BC}$ to point S. The bisectors of exterior angles ∠A and ∠C intersects at point I.

To Prove: The bisector of ∠B passes through I.

Construction: Draw perpendiculars $\overline{IL}$ and $\overline{IO}$ from point I on $\overline{BT}$ and $\overline{BS}$. Join points I and B.

Proof:

In $\Delta ILB \leftrightarrow \Delta IOB$	
$\overline{IL} \cong \overline{IO}$	Point I is equidistance of from $\overline{BT}$ and $\overline{BS}$.
$\overline{IB} \cong \overline{IB}$	Common
∴ $\angle LBI \cong \angle OBI$	Construction
∴ $\Delta ILB \cong \Delta IOB$	S.S.A ≅ S.S.A
Or $m\angle LBI = m\angle OBI$	Corresponding angles of congruent triangles.
$\overline{IB}$ is the bisector of ∠B	
∴ The bisector of two exterior angles	Due to congruency of angles.

Solved Review Exercise 12

1. Which of the following are true and which are false?

(i) Bisection means to divide into two equal parts ....

(ii) Right bisection of line segment means to draw perpendicular which passes through the mid-point of line segment. ....

(iii) Any point on the right bisector of a line segment is not equidistant from its end points. ....

(iv) Any point equidistant from the end points of a line segment is on the right bisector of it. ....

(v) The right bisectors of the sides of a triangle are not concurrent. ....

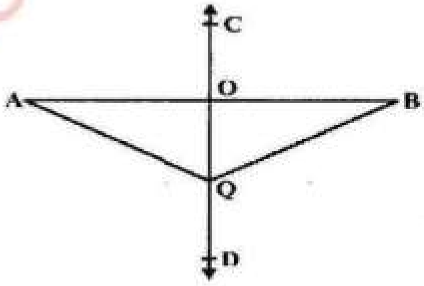
(vi) The bisectors of the angles of a triangle are concurrent. ....

(vii) Any point on the bisector of an angle is not equidistant from its arms. ....

(viii) Any point inside an angle, equidistant from its arms, is on the bisector of it. ....

Solution: (i) T (ii) T (iii) F (iv) T (v) F (vi) T (vii) F (viii) T

2. If $\overline{CD}$ is right bisector of line segment $\overline{AB}$, then



(i) $m\overline{OA}$ = ------

(ii) $m\overline{AQ}$ = ------

Solution:

(i) $m\overline{OB}$ (ii) $m\overline{BQ}$

3. Define the following

(i) Bisector of a line segment

(ii) Bisector of an angle

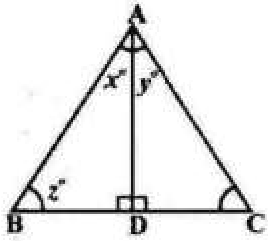
Solution: (i) Bisector of a line segment:

A line is called a right bisector of a line segment if 'l' is perpendicular to the line segment and passes through its mid points.

(ii) Bisector of an angle:

A ray BP is called the bisector of $\angle ABC$ if P is a point in the interior of the angle and $\angle ABP = \angle PBC$.

4. If the given triangle ABC is equilateral triangle and AD is bisector of angle A, then find the values of unknowns x°, y° and z°.



Solution:

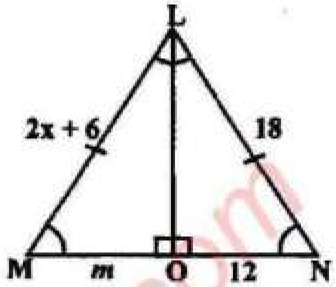
As ABC is an equilateral triangle, so

$\angle A = \angle B = \angle C = 60^\circ$

So, $x^\circ = 30^\circ$, $y^\circ = 30^\circ$

and $z^\circ = 60^\circ$

5. In the given congruent triangles LMO and LNO, find the unknowns x and m.



Solution: As ΔLMO and ΔLNO are congruent, so

$$\overline{LM} \cong \overline{LN}$$
$$2x + 6 = 18$$
$$2x = 18 - 6$$
$$2x = 12$$
$$\Rightarrow \quad x = 6$$

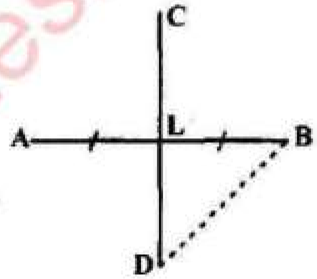
And $\quad \overline{MO} \cong \overline{ON}$

$$m = 12$$

6. $\overline{CD}$ is right bisector of the line segment AB.

(i) If $m\overline{AB} = 6$ cm, then find the $m\overline{AL}$ and $m\overline{LB}$.

(ii) If $m\overline{BD} = 4$ cm, then find $m\overline{AD}$.



Solution: (i) $\quad m\overline{AB} = 6m$

As point 'L' is the mid point of AB

So $\quad m\overline{AL} = \frac{1}{2}\overline{AB} = \frac{1}{2}(6) = 3cm$

$$m\overline{LB} = \frac{1}{2}\overline{AB} = \frac{1}{2}(6) = 3cm$$

(ii) As $\quad m\overline{BD} = 4cm$

But $\quad m\overline{AD} = m\overline{BD}$

So, $\quad m\overline{AD} = 4cm$

SUMMARY

We stated and proved the following theorems:

- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point equidistant from the end points of a line segment is on the right bisector of it.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on the bisector of it.
- The bisectors of the angles of a triangle are concurrent.
- Right bisection of a line segment means to draw a perpendicular at the midpoint of line segment.
- Bisection of an angle means to draw a ray to divide the given angle into two equal parts.

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UNIT 13

SIDES AND ANGLES OF A TRIANGLE

Unit Outlines
13.1 (i) *Sides of a Triangle* (ii) *Angles of a Triangle*

STUDENTS LEARNING OUTCOMES

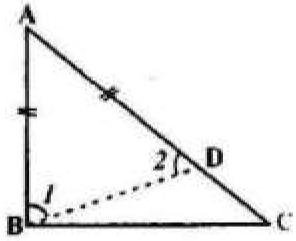
After studying this unit the students will be able to:

- prove that if two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- prove that if two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- prove that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- prove that from a point, out-side a line, the perpendicular is the shortest distance from the point to the line.

Introduction:

Recall that if two sides of a triangle are equal then the angles apposite to them are also equal and vice-versa. But in this unit we shall study some interesting inequality relations among sides and angles of a triangle.

Theorem 13.1.1: If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.



Solution: Given:

In ΔABC, $m\overline{AC} > m\overline{AB}$

To Prove:

$m\angle ABC > m\angle ACB$

Construction:

On $\overline{AC}$ take a point D such that $\overline{AD} \cong \overline{AB}$. Join B to D so that ΔADB is an isosceles triangle. Label $\angle 1$ and $\angle 2$ as shown in the given figure.

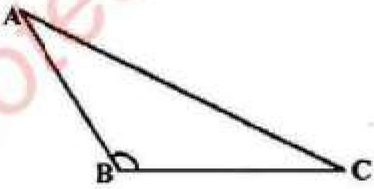
Proof:

Statements	Reasons
In ΔABD $m\angle 1 = m\angle 2$ ….. (i)	Angles opposite to congruent sides, (construction)
In ΔBCD, $m\angle ACB < m\angle 2$ i.e., $m\angle 2 > m\angle ACB$ ….. (ii)	An exterior angle of a triangle is greater than a

Statements	Reasons
	non-adjacent interior angle
$\therefore$ $m\angle 1 > m\angle ACB$ ...... (iii)	By (i) and (ii)
But	
$m\angle ABC = m\angle 1 + m\angle DBC$	Postulate of addition of angles.
$\therefore$ $m\angle ABC > m\angle 1$ ...... (iv)	
$\therefore$ $m\angle ABC > m\angle 1 > m\angle ACB$	By (iii) and (iv)
Hence $m\angle ABC > m\angle ACB$	(Transitive property of inequality of real numbers)

Example-1: Prove that in a scalene triangle, the angle opposite to the largest side is of measure greater than 60°. (i.e., two-third of a right-angle)

Solution: Given:



In ΔABC,

$m\overline{AC} > m\overline{AB}, m\overline{AC} > m\overline{BC}$

To Prove:

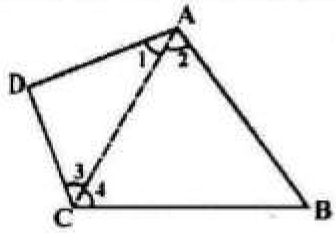
$m\angle B > 60°$

Proof:

Statements	Reasons
In ΔABC	
$m\angle B > m\angle C$	$m\overline{AC} > m\overline{AB}$ (given)
$m\angle B > m\angle A$	$m\overline{AC} > m\overline{BC}$ (given)
But $m\angle A + m\angle B + m\angle C = 180°$	$\angle A, \angle B, \angle C$ are the angles of ΔABC
$m\angle B + m\angle B + m\angle B > 180^0$	$m\angle B > m\angle C, m\angle B > m\angle A$ (proved)
Hence $m\angle B > 60°$	$180°/3 = 60°$

Example-2: In a quadrilateral ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side. Prove that $m\angle BCD > m\angle BAD$.

Solution: Given:



In quadrilateral ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side.

To Prove: $m\angle BCD > m\angle BAD$

Construction:

Joint A to C. Name the angles $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ as shown in the figure.

Proof:

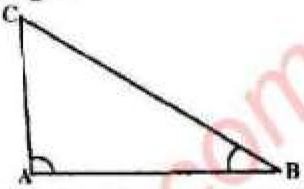
Statements	Reasons
In $\Delta ABC, m\angle 4 > \angle 2$ I	$m\overline{AB} > m\overline{BC}$ (given)
In $\Delta ACD, m\angle 3 > m\angle 1$ II	$m\overline{AD} > m\overline{CD}$ (given)
$\therefore m\angle 4 + m\angle 3 > m\angle 2 + m\angle 1$	From I and II
Hence $m\angle BCD > m\angle BAD$	$\because \begin{cases} m\angle 4 + m\angle 3 = m\angle BCD \\ m\angle 2 + m\angle 1 = m\angle BAD \end{cases}$

Theorem 13.1.2: (Converse of Theorem 13.1.1)

If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.

Solution: Given:

In ΔABC, $m\angle A > m\angle B$



To Prove:

$m\overline{BC} > m\overline{AC}$

Proof:

Statements	Reasons
If $m\overline{BC} \not> m\overline{AC}$, then	
either (i) $m\overline{BC} = m\overline{AC}$	
or (ii) $m\overline{BC} < m\overline{AC}$	(Trichotomy property of real numbers)
From (i) if $m\overline{BC} = m\overline{AC}$, then	
$m\angle A = m\angle B$	(Angles opposite to congruent sides are congruent)
Which is not possible	Contrary to the given.
From (ii) if $m\overline{BC} < m\overline{AC}$, then	
$m\angle A < m\angle B$	(The angle opposite to longer side is greater than angle opposite to smaller side)
This is also not possible.	Contrary to the given.
$\therefore$ $m\overline{BC} \neq m\overline{AC}$	
and $m\overline{BC} \not< m\overline{AC}$	Trichotomy property of real numbers.
Thus $m\overline{BC} > m\overline{AC}$	

Corollaries:

(i) The hypotenuse of a right angle triangle is longer than each of the other two sides.

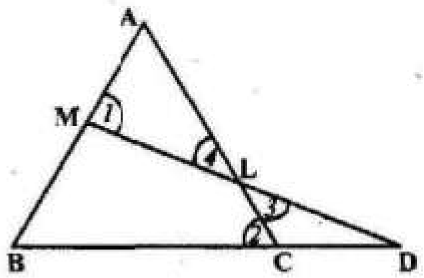
(ii) In an obtuse angled triangle, the side opposite- to the obtuse angle is longer than each of the other two sides.

Example: ABC is an isosceles triangle with base $\overline{BC}$. On $\overline{BC}$ a point D is taken away from C. A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M. Prove that $m\overline{AL} > m\overline{AM}$.

Solution: Given:

In ΔABC, $\overline{AB} \cong \overline{AC}$. D is a point on $\overline{BC}$ away from C.

A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M.



To Prove:

$m\overline{AL} > m\overline{AM}$

Proof:

Statements	Reasons
In ΔABC	
$\angle B \cong \angle 2$ I	$\overline{AB} \cong \overline{AC}$ (given)
In ΔMBD	
$m\angle 1 > m\angle B$ II	($\angle 1$ is an ext. $\angle$ and $\angle B$ is its internal opposite $\angle$)
$\therefore$ $m\angle 1 > m\angle 2$ III	From I and II
In ΔLCD,	
$m\angle 2 > m\angle 3$ IV	($\angle 2$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore$ $m\angle 1 > m\angle 3$ V	From III and IV
But $\angle 3 \cong \angle 4$ VI	Vertical angles
$\therefore$ $m\angle 1 > m\angle 4$	From V and VI
Hence $m\overline{AL} > m\overline{AM}$	In ΔALM, $m\angle 1 > m\angle 4$ (proved)

Theorem 13.1.3: The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Solution: Given:

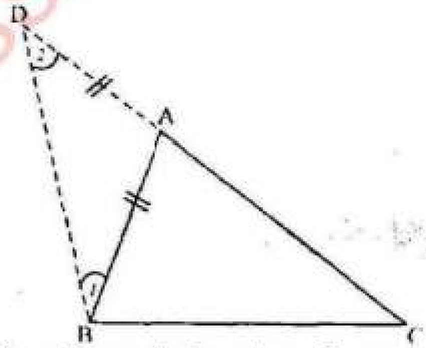
ΔABC

To Prove:

(i) $m\overline{AB} + m\overline{AC} > m\overline{BC}$

(ii) $m\overline{AB} + m\overline{BC} > m\overline{AC}$

(iii) $m\overline{BC} + m\overline{CA} > m\overline{AB}$



Construction:

Take a point D on $\overline{CA}$ such that $\overline{AD} \cong \overline{AB}$, Join B to D and name the angles $\angle 1$, $\angle 2$ as shown in the given figure.

Proof:

Statements	Reasons
In ΔABD,	
$\angle 1 \cong \angle 2$ ......(i)	$\overline{AD} \cong \overline{AB}$ (construction)
$m\angle DBC > m\angle 1$ ...... (ii)	$m\angle DBC = m\angle 1 + m\angle ABC$
$\therefore$ $m\angle DBC > m\angle 2$ ...... (iii)	From (i) and (ii)
In ΔDBC,	
$m\overline{CD} > m\overline{BC}$	By (iii)
i.e., $m\overline{AD} + m\overline{AC} > m\overline{BC}$	$m\overline{CD} = m\overline{AD} + m\overline{AC}$
Hence $m\overline{AB} + m\overline{AC} > m\overline{BC}$	$m\overline{AD} = m\overline{AB}$ (construction)
Similarly,	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	
and $m\overline{BC} + m\overline{CA} > m\overline{AB}$	

Example-1: Which of the following sets of lengths can be the lengths of the sides of a triangle?

(a) 2 cm, 3 cm, 5 cm **(b)** 3 cm, 4 cm, 5 cm, **(c)** 2 cm, 4 cm, 7 cm,

Solution:

(a) As $2+3=5$

$\therefore$ This set of lengths cannot be those of the Sides of a triangle.

(b) As $3+4>5, 3+5>4, 4+5>3$

$\therefore$ This set can form a triangle

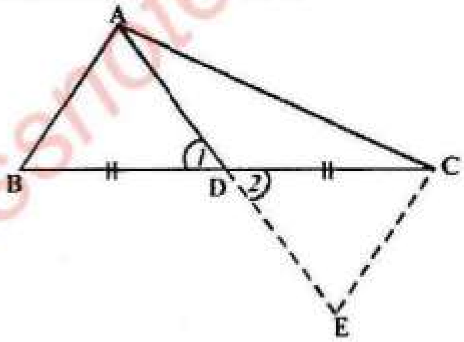
(c) As $2+4<7$

$\therefore$ This set of lengths cannot be the sides of a triangle.

Example-2: Prove that the sum of the measures of two sides of a triangle is greater than twice the measure of the median which bisects the third side.

Solution: Given:

In ΔABC, median AD bisects side $\overline{BC}$ at D.



To Prove:

$m\overline{AB}+m\overline{AC}>2m\overline{AD}$

Construction:

On $\overline{AD}$, take a point E, such that $\overline{DE}\cong\overline{AD}$. Join C to E. Name the angles $\angle 1$, $\angle 2$ as shown in the figure.

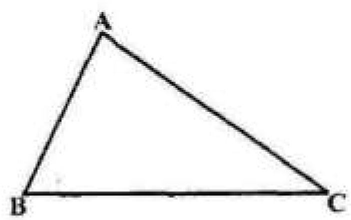
Proof:

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta ECD$	
$\overline{BD}\cong\overline{CD}$	Given
$\angle 1\cong\angle 2$	Vertical angles
$\overline{AD}\cong\overline{ED}$	Construction
$\Delta ABD\cong\Delta ECD$	S.A.S. Postulate
$\overline{AB}\cong\overline{EC}$ .... (i)	Corresponding sides of $\cong\Delta$s
$m\overline{AC}+m\overline{EC}>m\overline{AE}$ ......(ii)	ACE is a triangle
$m\overline{AC}+m\overline{AB}>m\overline{AE}$	From (i) and (ii)
Hence $m\overline{AC}+m\overline{AB}>2m\overline{AD}$	$m\overline{AE}=2m\overline{AD}$ (construction)

Example-3: Prove that the difference of measures of two sides of a triangle is less than the measure of the third side.

Solution: Given:

ΔABC



To Prove:

$m\overline{AC}-m\overline{AB}<m\overline{BC}$

$m\overline{BC}-m\overline{AB}<m\overline{AC}$

$m\overline{BC}-m\overline{AC}<m\overline{AB}$

Proof:

Statements	Reasons
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	ABC is a triangle
$(m\overline{AB} + m\overline{BC} - m\overline{AB}) > (m\overline{AC} - m\overline{AB})$	Subtracting $m\overline{AB}$ from both sides
$\therefore \quad m\overline{BC} > (m\overline{AC} - m\overline{AB})$	
or $\quad m\overline{AC} - m\overline{AB} < m\overline{BC}$ .....(i)	$a > b \Rightarrow b < a$
Similarly	
$\left.\begin{array}{l} m\overline{BC} - m\overline{AB} < m\overline{AC} \\ m\overline{BC} - m\overline{AC} < m\overline{AB} \end{array}\right\}$	Reason similar to (i)

Solved Exercise 13.1

1. Two sides of a triangle measure 10 cm and 15 cm. Which of the following measure is possible for the third side?

(a) 5 cm (b) 20 cm (c) 25 cm (d) 30 cm

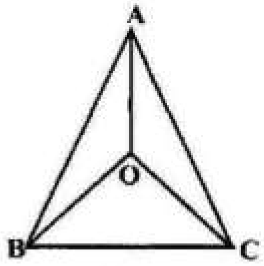
Solution: As the sum of measures of any two sides of a triangle is greater than the measure of the third side. So,

(a) $10 + 15 < 5$
(b) $10 + 15 < 20$
(c) $10 + 15 < 25$
(d) $10 + 15 < 30$

So, (d) 30 cm is correct.

2. "O" is an interior point of the Δ ABC. Show that

$$m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$$



Solution: Given:

O is any point inside the triangle ABC.

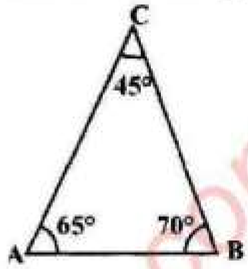
To prove:

$m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$.

Proof:

Statements	Reasons
In $\Delta AOB \quad m\overline{OA} + m\overline{OB} > m\overline{AB}$ ..... (i)	
In $\Delta AOC \quad m\overline{OA} + m\overline{OC} > m\overline{AC}$ ...... (ii)	
In $\Delta BOC \quad m\overline{OB} + m\overline{OC} > m\overline{BC}$ ...... (iii)	
Now	
$m\overline{OA} + m\overline{OB} + m\overline{OA} + m\overline{OC} + m\overline{OB} + m\overline{OC} > m\overline{AB} + m\overline{AC} + m\overline{BC}$	By adding (i), (ii) and (iii)
$2m\overline{OA} + 2m\overline{OB} + 2m\overline{OC} > m\overline{AB} + m\overline{AC} + m\overline{BC}$	
$2(m\overline{OA} + m\overline{OB} + m\overline{OC}) > m\overline{AB} + m\overline{AC} + m\overline{BC}$	
$m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{AC} + m\overline{BC})$	

3. In the ΔABC, $m\angle B = 70^\circ$ and $m\angle C = 45^\circ$. Which of the sides of the triangle is longest and which is the shortest?



Solution: Given:

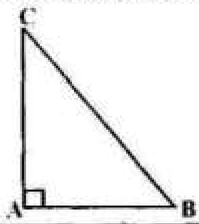
$m\angle B = 70^\circ$ and $m\angle C = 45^\circ$

In ΔABC

$$\begin{aligned} m\angle A &= 180^\circ - (m\angle B + m\angle C) \\ &= 180^\circ - (70^\circ + 45^\circ) \\ &= 180^\circ - 45^\circ \\ &= 65^\circ \end{aligned}$$

Longest side is $\overline{AC}$, because the side opposite to the greater angle is longer than the other sides. Shortest side is $\overline{AB}$, because the side opposite to the smaller angle is shortest then the other sides.

4. Prove that in a right-angled triangle, the hypotenuse is longer than each of the other two sides.



Solution: Given:

In Δ ABC, $m\angle A = 90^\circ$

To prove:

$m\overline{BC} > m\overline{AB}$ and $m\overline{BC} > m\overline{AC}$

Proof:

Statements	Reasons
If $m\overline{BC}$ is not greater than $m\overline{AB}$	
Then either $m\overline{BC} \cong m\overline{AB}$ ....(i) or $m\overline{BC} < m\overline{AB}$ ....(ii)	Trichotomy property
From (i) $m\overline{BC} \cong m\overline{AC}$ Then $m\angle A \cong m\angle C$ and $m\angle C = 90^\circ$	
Which is impossible in a triangle because there is any one right angle in a triangle.	$\because m\angle A = 90^\circ$
If $m\overline{BC}$ is not greater than $m\overline{AB}$	
Then $m\overline{BC} \cong m\overline{AC}$ ....(iii) or $m\overline{BC} < m\overline{AC}$ ....(iv)	Trichotomy property
From (iii) $m\overline{BC} \cong m\overline{AC}$ Then $m\angle A \cong m\angle B$ So, $m\angle B = 90^\circ$	$\because m\angle A = 90^\circ$
Which is impossible in a triangle because there is any one right angle in a triangle.	
If $m\overline{BC} < m\overline{AB}$ Then $m\angle A < m\angle C$ $90^\circ < m\angle C$	$\because m\angle A = 90^\circ$

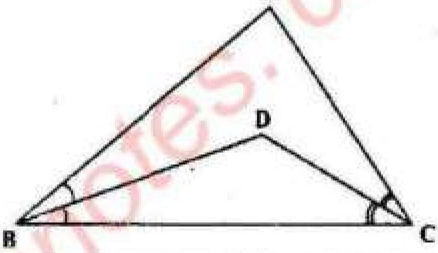
Statements	Reasons
Which is impossible in a triangle.	
Hence $m\overline{BC} > m\overline{AB}$	
Similarly, we can prove	
$m\overline{BC} > m\overline{AC}$	

5. In the triangular figure, $m\overline{AB} > m\overline{AC}$. $\overline{BD}$ and $\overline{CD}$ are the bisectors of $\angle B$ and $\angle C$ respectively. Prove that $m\overline{BD} > m\overline{DC}$.

Solution: Given:

$\overline{AB} > \overline{AC}$

$\overline{BD}$ and $\overline{CD}$ are the bisectors of $\angle B$ and $\angle C$ respectively.



To Prove:

$\overline{BD} > \overline{DC}$

Proof:

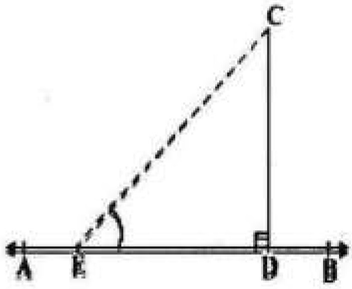
Statements		Reasons
In	$\Delta ABC \leftrightarrow \Delta BDC$	
	$\overline{BC} \cong \overline{BC}$	Common
	$\angle CBD \cong \angle CBA$	Congruent angles
	$\angle BCD \cong \angle BCA$	Congruent angels
	$\Delta ABC \cong \Delta BDC$	A.S.A postulate
AS	$\overline{AB} > \overline{AC}$	
So	$\overline{BD} > \overline{DC}$	Corresponding sides of congruent angles.

Theorem 13.1.4: From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Solution: Given:

A line AB and a point C (not lying on $\overleftrightarrow{AB}$) and a point D on $\overleftrightarrow{AB}$ such that

$\overline{CD} \perp \overleftrightarrow{AB}$.



To Prove:

$m\overline{CD}$ is the shortest distance from the point C to $\overleftrightarrow{AB}$.

Construction:

Take a pint E on $\overleftrightarrow{AB}$. Join C and E to form a ΔCDE.

Proof:

Statements	Reasons
In ΔCDE	
$m\angle CDB > m\angle CED$	(An exterior angle of a triangle is greater than non adjacent interior angle).

Statements	Reasons
But $m\angle CDB = m\angle CDE$	Supplement of right angle.
$\therefore$ $m\angle CDE > m\angle CED$	
or $m\angle CED < m\angle CDE$	$a > b \Rightarrow b < a$
or $m\overline{CD} < m\overline{CE}$	Side opposite to greater angle is greater.
But E is any point on AB	
Hence $m\overline{CD}$ is the shortest distance from C to $\overline{AB}$.	

Note:

(i) The distance between a line and a point not on it, is the length of the perpendicular line segment from the point to the line.

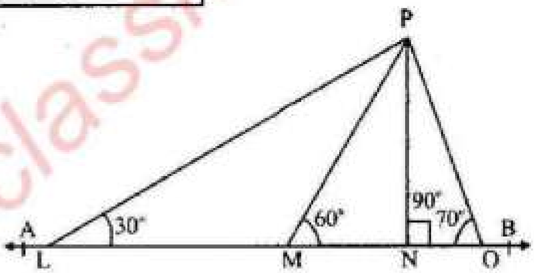
(ii) The distance between a line and a point lying on it is zero.

Solve Exercise 13.2

1. In the figure, P is any point and AB is a line. Which of the following is the shortest distance between the point P and the line AB?

(a) $m\overline{PL}$ (b) $m\overline{PM}$

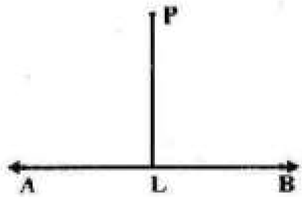
(c) $m\overline{PN}$ (d) $m\overline{PO}$



Solution:

(c) $m\ \overline{PN}$; because from a point outside a line, the perpendicular distance is the shortest distance.

2. In the figure, P is any point lying away from the line AB. Then $m\overline{PL}$ will be the shortest distance if



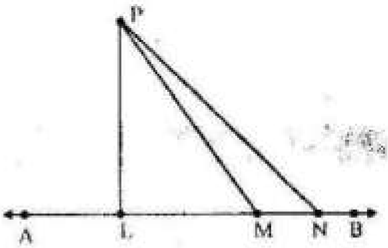
(a) $m\angle PLA = 80°$

(b) $m\angle PLB = 100°$

(c) $m\angle PLA = 90°$

Solution: (c) $m\ \angle PLA = 90°$ because the perpendicular distance is the shortest distance from the point to the line.

3. In the figure, $\overline{PL}$ is perpendicular to the line AB and $m\overline{LN} > m\overline{LM}$. Prove that $m\overline{PN} > m\overline{PM}$.



Solution:

Given:

$\overline{PL} \perp \overline{AB}$ and $m\overline{LN} > m\overline{LM}$

To prove:

$m\overline{PN} > m\overline{PM}$

Proof:

Statements	Reasons
In ΔPLM and ΔPLN $\overline{PL} \sim \overline{PL}$ $(\overline{PL})^2 + (\overline{LN})^2 > (\overline{PL})^2 + (\overline{LM})^2$ $(\overline{PN})^2 > (\overline{PM})^2$ $m\overline{PN} > m\overline{PM}$	

REVIEW EXERCISE 13

1. Which of the following are true and which are false?

(i) The angle opposite to the longer side of a triangle is greater. ....

(ii) In a right-angled triangle greater angle is of 60°. ....

(iii) In an isosceles right-angled triangle, angles other than right angle are each of 45°. ....

(iv) A triangle having two congruent sides is called equilateral triangle ....

(v) A perpendicular from a point to a line is shortest distance. ....

(vi) Perpendicular to line form an angle of 90°. ....

(vii) A point out side the line is collinear with it. ....

(viii) Sum of two sides of triangle is greater than the third. ....

(ix) The distance between a line and a point on it is zero. ....

(x) Triangle can be formed of lengths 2 cm, 3 cm and 5 cm. ....

Solution: (i) T (ii) F (iii) T (iv) F (v) T (vi) T
(vii) F (viii) T (ix) T (x) F

2. What will be angle for shortest distance from an outside point to the line?

Solution:

The angle for shortest distance from an outside point to the line is 90°.

3. If 13 cm, 12 cm, and 5 cm are the lengths of a triangle, then verify that difference of measures of any two sides of a triangle is less than the measure of the third side.

Solution: As the difference of measures of any two sides of a triangle is less than the measure of the third side. So,

$$15 - 13 < 5 \quad \Rightarrow \quad 2 < 5,$$
$$12 - 5 < 13 \quad \Rightarrow \quad 7 < 13,$$
$$13 - 12 < 5 \quad \Rightarrow \quad 1 < 5$$

4. If 10 cm, 6 cm and 8 cm are the lengths of a triangle, then verify that sum of measures of two sides of a triangle is greater than the third side.

Solution: As the sum of measure of any two sides of a triangle is greater than the measure of the third side. So,

$6 + 8 > 10 \Rightarrow 14 > 10,$
$10 + 6 > 8 \Rightarrow 16 > 8,$
$10 + 8 > 6 \Rightarrow 18 > 6$

5. 3 cm, 4 cm and 7 cm are not the lengths of the triangle. Give the reason.

Solution: As the sum of measure of any two sides of a triangle is greater than the measure of the third side. But

$3 + 4 \not> 7$

So, given measurements are not the lengths of a triangle.

6. If 3 cm and 4 cm are lengths of two sides of a right angle triangle then what should be the third length of the triangle.

Solution: Let a = 3, b = 4, c = ?

$c^2 = a^2 + b^2$

$c^2 = 3^2 + 4^2 = 9 + 16 = 25$

$\Rightarrow c = 5$

SUMMARY

We stated and proved the following theorems:

- If two sides of a triangle are unequal in length,, the longer side has an angle of greater measure opposite to it.
- If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

UNIT 14

RATIO AND PROPORTION

Unit Outlines

14.1 *Ratio and Proportion*

STUDENTS LEARNING OUTCOMES

After studying this unit, the students will be able to:

- prove that a line parallel to one side of a triangle, intersecting the other two sides, divides them proportionally.
- prove that if a line segment intersects the two sides of a triangle in the same ratio, then it is parallel to the third side.
- prove that the internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.
- prove that if two triangles are similar, the measures of their corresponding sides are proportional.

Introduction:

We will prove some theorems and corollaries involving ratio and proportions of sides of triangle and similarity of triangles. Knowledge of ratio and proportion is necessary requirement of many occupations like food service occupation, medications in health, preparing maps for land survey and construction works, profit to cost ratios etc.

Ratio:

Ratio $a : b = \frac{a}{b}$ is the comparison of two alike quantities a and b, called the elements of a ratio.

Proportion:

Equality of two ratios is defined as proportion. That is, if $a : b = c : d$, then a, b, c and d are said to be in proportion.

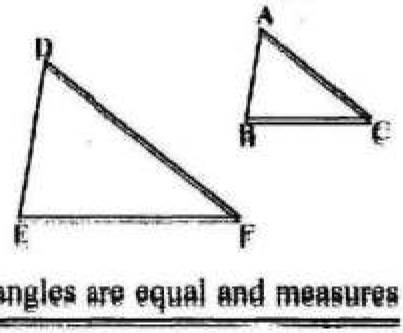
Similar Triangles:

Equally important are the similar shapes. In particular the similar triangles that have many practical applications. Geometrical figures can also be similar, e.g., if

In $\Delta ABC \leftrightarrow \Delta DEF$

$\angle A \cong \angle D, \quad \angle B \cong \angle E, \quad \angle C \cong \angle F$

and $\quad \frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{CA}}{\overline{FD}}$



Then ΔABC and ΔDEF are called similar triangles which is symbolically written as

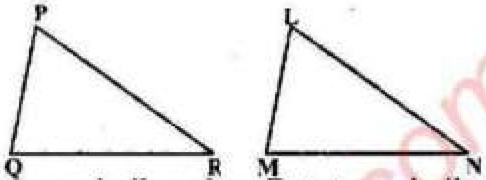
$\Delta ABC \sim \Delta DEF$

It means that corresponding angles of similar triangles are equal and measures

of their corresponding sides are proportional.

$\Delta PQR \cong \Delta LMN$ means that in $\Delta PQR \leftrightarrow \Delta LMN$

$\angle P \cong \angle L$, $\angle Q \cong \angle M$, $\angle R \cong \angle N$,
$\overline{PQ} \cong \overline{LM}$, $\overline{QR} \cong \overline{MN}$, $\overline{RP} \cong \overline{NL}$

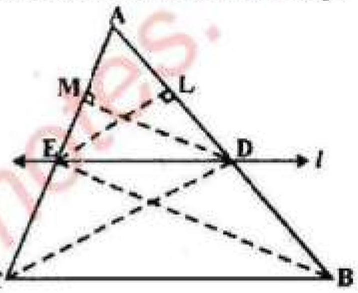


Now as $\dfrac{\overline{PQ}}{\overline{LM}} = \dfrac{\overline{QR}}{\overline{MN}} = \dfrac{\overline{RP}}{\overline{NL}} = 1$

$\therefore$ $\Delta PQR \sim \Delta LMN$

In other words, two congruent triangles are similar also. But two similar triangles are not necessarily congruent, as congruence of their sides is not necessary.

Theorem 14.1.1: A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.



Solution: Given:

In ΔABC, the line l is intersecting the sides $\overline{AC}$ and $\overline{AB}$ at points E and D respectively such that $\overline{ED} \parallel \overline{CB}$.

To Prove: $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

Construction:

Join B to E and C to D. From D draw $\overline{DM} \perp \overline{AC}$ and from E draw $\overline{EL} \perp \overline{AB}$.

Proof:

Statements	Reasons
In triangles BED and AED, $\overline{EL}$ is the common perpendicular.	
$\therefore$ Area of $\Delta BED = \frac{1}{2} \times m\overline{BD} \times m\overline{EL}$ ...(i)	Area of a $\Delta = \frac{1}{2}$ (base) (height)
And area of $\Delta AED = \frac{1}{2} \times m\overline{AD} \times m\overline{EL}$ .. (ii)	
Thus $\dfrac{\text{Area of } \Delta BED}{\text{Area of } \Delta AED} = \dfrac{m\overline{DB}}{m\overline{AD}}$ .. (iii)	Dividing (i) by (ii)
Similarly $\dfrac{\text{Area of } \Delta CDE}{\text{Area of } \Delta ADE} = \dfrac{m\overline{EC}}{m\overline{AE}}$ .. (iv)	
But $\Delta BED \cong \Delta CDE$	(Areas of triangles with common base and same altitudes are equal. Given that $\overline{ED} \mid \overline{CB}$, so altitudes are equal.
$\therefore$ From (iii) and (iv), we have $\dfrac{m\overline{DB}}{m\overline{AD}} = \dfrac{m\overline{EC}}{m\overline{AE}}$ or $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$	Taking reciprocal of both sides.
Hence $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$	

Observe that:

From the above theorem we also have

$$\frac{m\overline{BD}}{m\overline{AB}}=\frac{m\overline{CE}}{m\overline{AC}} \text{ and } \frac{m\overline{AD}}{m\overline{AB}}=\frac{m\overline{AE}}{m\overline{AC}}$$

Corollaries:

(a) If $\frac{m\overline{AD}}{m\overline{AB}}=\frac{m\overline{AE}}{m\overline{AC}}$, then $\overline{DE}\parallel\overline{BC}$

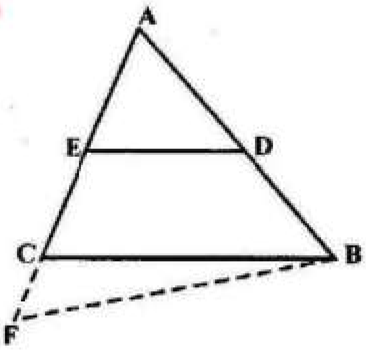
(b) If $\frac{m\overline{AB}}{m\overline{DB}}=\frac{m\overline{AC}}{m\overline{EC}}$, then $\overline{DE}\parallel\overline{BC}$

Points to be noted:

(i) Two points determine a line and three non-collinear points determine a plane.
(ii) A line segment has exactly one midpoint.
(iii) If two intersecting lines form equal adjacent angles, the lines are perpendicular.

Theorem 14.1.2: (Converse of Theorem 14.1.1)

If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.



Solution: Given:

In ΔABC, $\overline{ED}$ intersects $\overline{AB}$ and $\overline{AC}$. Such that $m\overline{AD}:m\overline{DB}=m\overline{AE}:m\overline{EC}$

To Prove:

$\overline{ED}\parallel\overline{CB}$

Construction:

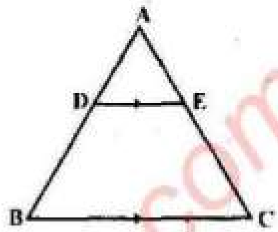
If $\overline{ED}\nparallel\overline{CB}$, then draw $\overline{BF}\parallel\overline{DE}$ to meet $\overline{AC}$ produced at F.

Proof:

Statements	Reasons
In ΔABF	
$\overline{DE}\parallel\overline{BF}$	Construction
$\therefore$ $\frac{m\overline{AD}}{m\overline{DB}}=\frac{m\overline{AE}}{m\overline{EF}}$ .... (i)	(A line parallel to one side of a triangle divides the other two sides proportionally)
But $\frac{m\overline{AD}}{m\overline{DB}}=\frac{m\overline{AE}}{m\overline{EC}}$ .... (ii)	Given
$\therefore$ $\frac{m\overline{AE}}{m\overline{EF}}=\frac{m\overline{AE}}{m\overline{EC}}$	From (i) and (ii)
or $m\overline{EF}=m\overline{EC}$	
Which is possible only if point F is coincident with C	(Property of real numbers.)
$\therefore$ Our supposition is wrong	
Hence $\overline{ED}\parallel\overline{CB}$	

Solved Exercise 14.1

1. In ΔABC, $\overline{DE} \parallel \overline{BC}$



(i) If $m\overline{AD} = 1.5$ cm, $m\overline{BD} = 3$ cm, $m\overline{AE} = 1.3$cm, then find $m\overline{CE}$.

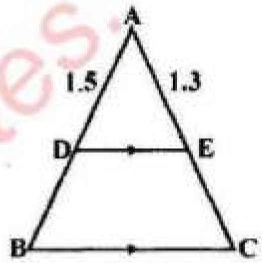
Solution: Given:

$$m\overline{AD} = 1.5\text{cm}, m\overline{BD} = 3\text{cm}, m\overline{AE} = 1.3\text{cm}$$

To find:

$$m\overline{CE} = ?$$

We know that



$$\frac{m\overline{AE}}{m\overline{CE}} = \frac{m\overline{AD}}{m\overline{BD}}$$

$$\frac{1.3}{m\overline{EC}} = \frac{1.5}{3}$$

$$m\overline{CE}(1.5) = (3)(1.3)$$

$$m\overline{EC} = \frac{(3)(1.3)}{1.5} = 2.6\text{cm}$$

(ii) If $m\overline{AD} = 2.4$ cm, $m\overline{AE} = 3.2$ cm, $m\overline{EC} = 4.8$cm, find $m\overline{AB}$.

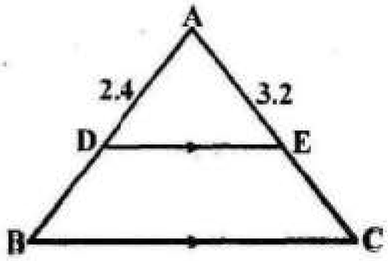
Solution: Given:

$$m\overline{AD} = 2.4\text{cm}, m\overline{AE} = 3.2\text{cm},$$

$$m\overline{EC} = 4.8\text{cm},$$

To find:

$$m\overline{AB} = ?$$



We know that

$$\frac{m\overline{AE}}{m\overline{EC}} = \frac{m\overline{AD}}{m\overline{BD}}$$

$$\frac{3.2}{4.8} = \frac{2.4}{m\overline{DB}}$$

$$m\overline{DB}(3.2) = (4.8)(2.4)$$

$$m\overline{DB} = \frac{(4.8)(2.4)}{3.2} = 3.6\text{cm}$$

$$m\overline{AB} = m\overline{AD} + m\overline{BD}$$

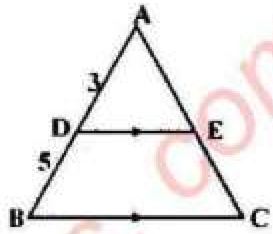
$$= 2.4 + 3.6$$

$$= 6 \text{ cm}$$

(iii) If $\frac{m\overline{AD}}{m\overline{DB}}=\frac{3}{5}$ and $m\overline{AC}=4.8cm$, find $m\overline{AE}$.

Solution: Given:

$$\frac{m\overline{AD}}{m\overline{DB}}=\frac{3}{5} \text{ and } m\overline{AC}=4.8cm$$



To find: $m\overline{AE}=?$

We know that:

$$\frac{m\overline{AD}}{m\overline{DB}}=\frac{m\overline{AE}}{m\overline{EC}}$$

$$\frac{3}{5}=\frac{m\overline{AE}}{m\overline{AC}-m\overline{AE}}$$

$$\frac{3}{5}=\frac{m\overline{AE}}{4.8-m\overline{AE}}$$

$$5\left(m\overline{AE}\right)=3\left(4.8-m\overline{AE}\right)$$

$$5\left(m\overline{AE}\right)=14.4-3\left(m\overline{AE}\right)$$

Adding 3(mAE) both sides we get

$$5\left(m\overline{AE}\right)+3\left(m\overline{AE}\right)=14.4-3\left(m\overline{AE}\right)+3\left(m\overline{AE}\right)$$

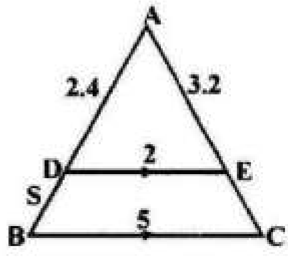
$$8\left(m\overline{AE}\right)=14.4$$

$$m\overline{AE}=1.8cm$$

(iv) If $m\overline{AD}=2.4cm$, $m\overline{AE}=3.2cm$, $m\overline{DE}=2cm$, $m\overline{BC}=5cm$, find $m\overline{AB}, m\overline{DB}, m\overline{AC}, m\overline{CE}$.

Solution: Given:

$$m\overline{AD}=2.4cm, m\overline{AE}=3.2cm,$$
$$m\overline{DE}=2cm, m\overline{BC}=5cm.$$



To find:

$$m\overline{AE}=?, m\overline{DB}=?$$
$$m\overline{AC}=?, m\overline{CE}=?$$

We know that

$$\frac{m\overline{AD}}{m\overline{BD}}=\frac{m\overline{AE}}{m\overline{CE}}$$

$$\frac{2.4}{m\overline{BD}}=\frac{3.2}{m\overline{CE}}$$

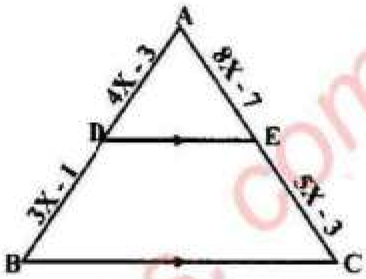
$$\Rightarrow \quad m\overline{CE}=3.2 \text{ and } m\overline{BD}=2.4$$

Now $m\overline{AB}=m\overline{AD}+m\overline{DB}=2.4+2.4=4.8cm$

And $m\overline{AC} = m\overline{AE} + m\overline{CE} = 3.2 + 3.2 = 6.4\text{cm}$

(v) If AD = 4x − 3, AE = 8x − 7, BD = 3x − 1, and CE = 5x − 3, find the value of x.

Solution: Given:



$m\overline{AD} = 4x - 3, m\overline{AE} = 8x - 7,$

$m\overline{BD} = 3x - 1, m\overline{CE} = 5x - 3.$

To find: $x = ?$

We know that

$$\frac{m\overline{AD}}{m\overline{BD}} = \frac{m\overline{AE}}{m\overline{CE}}$$

$$\frac{4x-3}{3x-1} = \frac{8x-7}{5x-3}$$

$$(3x-1)(8x-7) = (4x-3)(5x-3)$$

$$24x^2 - 21x - 8x + 7 = 20x^2 - 12x - 15x + 9$$

$$24x^2 - 29x + 7 = 20x^2 - 27x + 9$$

$$24x^2 - 29x + 7 - 20x^2 + 27x - 9 = 0$$

$$24x^2 - 20x^2 - 29x + 27x + 7 - 9 = 0$$

$$4x^2 - 2x - 2 \quad = 9$$

$$2(2x^2 - x - 1) \quad = 0$$

$$\Rightarrow \quad 2x^2 - 2x + x - 1 = 0$$

$$2x(x-1) + 1(x-1) = 0$$

$$(x-1)(2x+1) = 0$$

$$\Rightarrow \quad x - 1 = 0 \quad \text{or} \quad 2x + 1 = 0$$

$$x = 1 \qquad 2x = -1$$

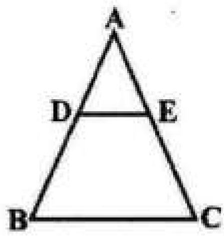
$$x = -\frac{1}{2}(\text{not possible})$$

So, $x = 1$

2. If ΔABC is an isosceles triangle, $\angle A$ is vertex angle and $\overline{DE}$ intersects the sides $\overline{AB}$ and $\overline{AC}$ as shown in the following figure so that

$m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

Prove that ΔADE is also an isosceles triangle.



Solution:

In ΔABC $\quad m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

To Prove:

In ΔADE $\quad m\overline{AD} = m\overline{AE}$

Let $m\overline{AD} = a, m\overline{AE} = b$ and $m\overline{AB} = m\overline{AC} = x$

Therefore

$\overline{BD} = x - a$ and $\overline{EC} = x - b$

Now $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$$

$$\frac{a}{x-a} = \frac{b}{x-b}$$

$$a(x-b) = b(x-a)$$

$$ax - ab = bx - ab$$

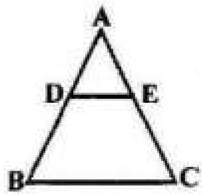
$$ax = bx$$

$$\Rightarrow \quad a = b$$

$$m\overline{AD} = m\overline{AE}$$

Therefore, ΔADE is an isosceles triangle.

3. In a equilateral triangle ABC shown in the following figure.
$m\overline{AE} : m\overline{AC} = m\overline{AD} : m\overline{AB}$
Find all the three angles of ΔADE and name it also.



Solution: Given:

ABC is an equilateral triangle.

$m\overline{BC} = m\overline{CA} = m\overline{AB}$

$\Rightarrow \quad m\overline{BC} = m\overline{CA}$ and $m\overline{BC} = m\overline{AB}$ and $m\overline{CA} = m\overline{AB}$

$\therefore \quad m\angle A = m\angle B$ (Opposite angles of equal sides)

And $m\angle B = m\angle C \Rightarrow m\angle A = m\angle C$

We know that

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

$$m\angle A + m\angle A + m\angle A = 180^\circ$$

$$3m\angle A = 180^\circ$$

$$m\angle A = 60^\circ$$

$\Rightarrow \quad m\angle A = m\angle B = m\angle C = 60^\circ$

4. Prove that the line segment drawn through the mid-point of one side of a triangle and parallel to another side bisects the third side.

Solution:

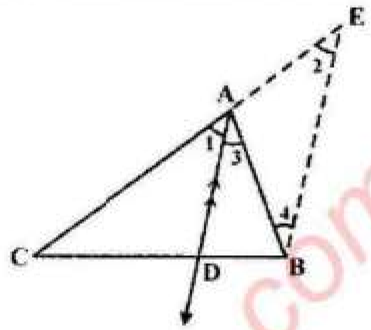
Same as theorem 14.1.1

5. Prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side.

Solution:

Same as theorem 14.1.2

Theorem 14.1.3: The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.



Given

In ΔABC internal angle bisector of $\angle A$ meets $\overline{CB}$ at the point D.

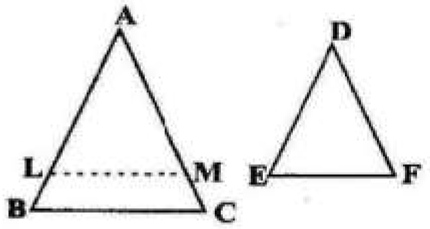
To Prove: $m\overline{BD} : m\overline{DC} = m\overline{AB} : m\overline{AC}$

Construction: Draw a line segment $\overline{BE} \parallel \overline{DA}$ to meet $\overline{CA}$ produced at E.

Proof:

Statements	Reasons
$\because \overline{AD} \parallel \overline{EB}$ and $\overline{EC}$ intersects them,	Construction
$\therefore m\angle 1 = m\angle 2$ .... (i)	Corresponding angles
Again $\overline{AD} \parallel \overline{EB}$	
And $\overline{AB}$ intersects them,	
$\therefore m\angle 3 = m\angle 4$ .... (ii)	Alternate angles
But $m\angle 1 = m\angle 3$	Given
$\therefore m\angle 2 = m\angle 4$	From (i) and (ii)
And $\overline{AB} \cong \overline{AE}$ or $\overline{AE} \cong \overline{AB}$	In a Δ, the sides opposite to congruent angles are also congruent.
Now $\overline{AD} \parallel \overline{EB}$	Construction
$\therefore \dfrac{m\overline{AD}}{m\overline{BC}} = \dfrac{m\overline{EA}}{m\overline{AC}}$	By Theorem 141.1
or $\therefore \dfrac{m\overline{BD}}{m\overline{DC}} = \dfrac{m\overline{AB}}{m\overline{AC}}$	$m\overline{EA} = m\overline{AB}$ (proved)
Thus $m\overline{BD} : m\overline{DC} = m\overline{AB} : m\overline{AC}$	

Theorem 14.1.4: If two triangles are similar, then the measures of their corresponding sides are proportional.



Solution: Given

$\Delta ABC \sim \Delta DEF$

i.e., $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$

To Prove: $\dfrac{m\overline{AB}}{m\overline{DE}} = \dfrac{m\overline{AC}}{m\overline{DF}} = \dfrac{m\overline{BC}}{m\overline{EF}}$

Construction: (I) Suppose that $m\overline{AB} > m\overline{DE}$

(II) $m\overline{AB} \leq m\overline{DE}$

On $\overline{AB}$ take a point L such that $m\overline{AL} = m\overline{DE}$.

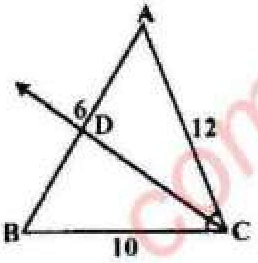
On $\overline{AC}$ take a point M such that $m\overline{AM} = m\overline{DF}$. Join L and M by the line

segment LM.

Proof:

Statements	Reasons
(I) In $\Delta ALM \leftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\overline{AL} \cong \overline{DE}$	Construction
$\overline{AM} \cong \overline{DF}$	Construction
Thus $\Delta ALM \cong \Delta DEF$	S.A.S. Postulate
and $\angle L \cong \angle E$, $\angle M \cong \angle F$	(Corresponding angles of congruent triangles)
Now $\angle E \cong \angle B$ and $\angle F \cong \angle C$	Given
$\therefore$ $\angle L \cong \angle B$ $\angle M \cong \angle C$	Transitivity of congruence
Thus $\overline{LM} \parallel \overline{BC}$	Corresponding angles are equal.
Hence $\dfrac{m\overline{AL}}{m\overline{AB}} = \dfrac{m\overline{AM}}{m\overline{AC}}$ ......(i)	By Theorem 14.1.1
or $\dfrac{m\overline{DE}}{m\overline{AB}} = \dfrac{m\overline{DF}}{m\overline{AC}}$	$m\overline{AL} = m\overline{DE}$ and $m\overline{AM} = m\overline{DF}$ (construction)
Similarly by intercepting segments on $\overline{BA}$ and $\overline{BC}$, we can prove that $\dfrac{m\overline{DE}}{m\overline{AB}} = \dfrac{m\overline{EF}}{m\overline{BC}}$ .....(ii)	
Thus $\dfrac{m\overline{DE}}{m\overline{AB}} = \dfrac{m\overline{DF}}{m\overline{AC}} = \dfrac{m\overline{EF}}{m\overline{BC}}$	by (i) and (ii)
or $\dfrac{m\overline{AB}}{m\overline{DE}} = \dfrac{m\overline{AC}}{m\overline{DF}} = \dfrac{m\overline{BC}}{m\overline{EF}}$	by taking treciprocals
(II) If $m\overline{AB} < m\overline{DE}$, it can similarly be proved by taking intercepts on the sides of ΔDEF.	
If $m\overline{AB} = m\overline{DE}$,	
Then in $\Delta ABC \leftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\angle B \cong \angle E$	Given
and $m\overline{AB} \cong m\overline{DE}$	
so $\Delta ABC \cong \Delta DEF$	A.S.A.≅A.S.A
Thus $\dfrac{m\overline{AB}}{m\overline{DE}} = \dfrac{m\overline{AC}}{m\overline{DF}} = \dfrac{m\overline{BC}}{m\overline{EF}} = 1$	$\overline{AC} \cong \overline{DF}, \overline{BC} \cong \overline{EF}$
Hence the result is true for all the cases.	

Solved Exercise 14.2



1. In ΔABC as shown in the figure, $\overline{CD}$ bisects $\angle C$ and meets $\overline{AB}$ at D. $m\overline{BD}$ is equal to

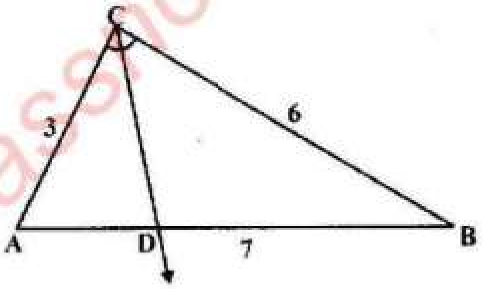
(a) 5 (b) 16 (c) 10 (d) 18

Solution: We know that

$$\frac{m\overline{BD}}{m\overline{AD}} = \frac{m\overline{BC}}{m\overline{AC}}$$

$$\frac{m\overline{BD}}{6} = \frac{10}{12}$$

$$m\overline{BD} = \frac{6\times 10}{12} = 5 \text{ units}$$



2. In ΔABC shown in the figure, $\overline{CD}$ bisects $\angle C$. If $m\overline{AC} = 3$, $m\overline{CB} = 6$ and $m\overline{AB} = 7$, then find $m\overline{AD}$ and $m\overline{DB}$.

Solution: $m\overline{AD} + m\overline{DB} = m\overline{AB}$

$$m\overline{DB} = m\overline{AB} - m\overline{AD}$$

$$m\overline{DB} = 7 - m\overline{AD} \quad \text{\_\_\_\_\_\_(1)}$$

We know that

$$\frac{m\overline{DB}}{m\overline{CB}} = \frac{m\overline{AD}}{m\overline{AC}}$$

$$\frac{7 - m\overline{AD}}{6} = \frac{m\overline{AD}}{3}$$

$$6m\overline{AD} = 3\left(7 - m\overline{AD}\right)$$

$$6m\overline{AD} = 21 - 3m\overline{AD}$$

$$6m\overline{AD} + 3m\overline{AD} = 21$$

$$9m\overline{AD} = 21$$

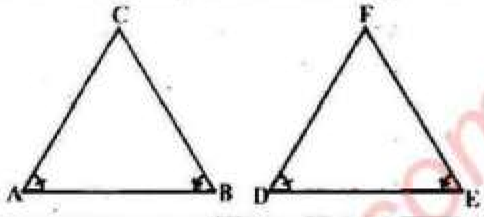
$$m\overline{AD} = \frac{21}{9} = \frac{7}{3} \text{units}$$

Put $m\overline{AD} = \frac{7}{3}$ in eq. (1), we get

$$m\overline{BD} = 7 - \frac{7}{3} = \frac{21-7}{3}$$

$$= \frac{14}{3} \text{units}$$

3. Show that in any correspondence of two triangles, if two angles of one triangle are congruent to the corresponding angles of the other, then the triangles are similar.



Solution: Given:

Two triangles ABC and DEF.

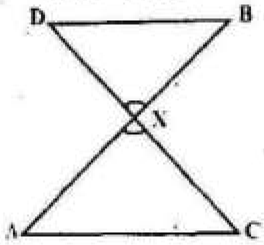
Where, $m\angle A = m\angle D$ and $m\angle B = m\angle E$

To prove: $\Delta ABC \sim \Delta DEF$

Proof:

Statements	Reasons
In ΔABC	
$m\angle A + m\angle B + m\angle C = 180^\circ$ ___(i)	Sum of all the angles of a triangle is 180°
In ΔDEF	
$m\angle D + m\angle E + m\angle E = 180^\circ$ ___(ii)	Sum of all the angles of a triangle is 180°
$m\angle A + m\angle B + m\angle C = m\angle D + m\angle E + m\angle F$	Combining eq. (i) and eq.(ii)
$m\angle A + m\angle B + m\angle C = m\angle A + m\angle B + m\angle F$	$\because m\angle A = m\angle D$ and $m\angle B = m\angle E$
$m\angle C = m\angle F$	
Because all the angles of both triangles are same:	
Hence ΔABC and ΔDEF are similar.	

4. If line segments AB and CD are intersecting at point X and $\frac{m\overline{AX}}{m\overline{XB}} = \frac{m\overline{CX}}{m\overline{XD}}$, then show that ΔAXC and ΔBXD are similar.



Solution:

Given:

Two lines $\overline{AB}$ and $\overline{CD}$ intersect each other at point x.

To Prove:

$\Delta AXC \cong \Delta DXB$

Proof:

Statements	Reasons
$\Delta AXC \leftrightarrow \Delta DXB$	
$\angle AXC = \angle DXB$	Vertical angles
$\therefore$ $\overline{AC} = \overline{DB}$	Opposite sides of similar angles
$\angle XAC = \angle XBD$	
and $\angle XCA = \angle XDB$	
$\therefore$ $\Delta AXC = \Delta DXB$	A.S.A Postulate

OOOOOOOOOOOOO

Solved Review Exercise 14

1. Which of the following are true and which are false?

(i) Congruent triangles are of same size and shape. …………
(ii) Similar triangles are of same shape but different sizes. …………
(iii) Symbol used for congruent is '≅'. …………
(iv) Symbol used for similarity is '~'. …………
(v) Congruent triangles are similar. …………
(vi) Similar triangles are congruent. …………
(vii) A line segment has only one mid point. …………
(viii) One and only one line can be drawn through two points. …………
(ix) Proportion is non-equality of two ratios. …………
(x) Ratio has no unit.

Solution: (i) T (ii) T (iii) T (iv) T (v) T
(vi) F (vii) T (viii) T (ix) F (x) T

2. Define the following:

(i) **Ratio** (ii) **Proportion**
(iii) **Congruent Triangles** (iv) **Similar Triangles**

Solution: (i) Ratio:

Comparison of two alike terms is called ratio.

(ii) Proportion:

Equality of two ratios is called proportion.

(iii) Congruent Triangles:

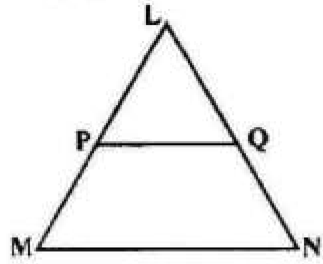
If the measures of corresponding sides are proportional then the triangles are called congruent triangles.

(iv) Similar Triangles:

If the measure of corresponding sides are equiangular and proportional then the triangles are called similar triangles.

3. In ΔLMN shown in the figure, $\overline{MN} \parallel \overline{PQ}$

(i) **If $m\overline{LM} = 5cm, m\overline{LP} = 2.5cm, m\overline{LQ} = 2.3cm$, then find $m\overline{LN}$.**
(ii) **If $m\overline{LM} = 6cm, m\overline{LQ} = 2.5cm, m\overline{QN} = 5cm$, then find $m\overline{LP}$.**



Solution: Given:

$$m\overline{LM} = 5cm, m\overline{LP} = 2.5cm, m\overline{LQ} = 2.3cm.$$

To find: (i) $m\overline{LN} = ?$

We know that

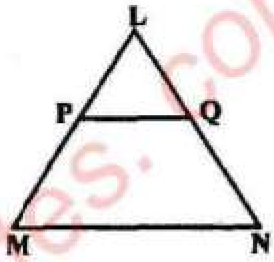
$$\frac{m\overline{LM}}{m\overline{LP}} = \frac{m\overline{LN}}{m\overline{LQ}}$$

$$\frac{5}{2.5}=\frac{m\overline{LN}}{2.3}$$

$$2.5m\overline{LN}=(5)(2.3)$$

$$m\overline{LN}=\frac{(5)(2.3)}{2.5}=4.6\text{cm}$$

(ii) $m\overline{LM}=6\text{cm}, m\overline{LQ}=2.5\text{cm}, m\overline{QN}=5\text{cm}, m\overline{LP}=?$



As, $m\overline{LN} = m\overline{LQ}+m\overline{QN}$

$= 2.5+5$

$m\overline{LN} = 7.5\text{cm}$

As we know that,

$$\frac{m\overline{LN}}{m\overline{LP}}=\frac{m\overline{LN}}{m\overline{LQ}}$$

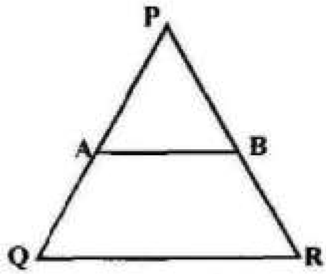
$$\frac{6}{m\overline{LP}}=\frac{7.5}{2.5}$$

$$(7.5)(m\overline{LP})=6\times 2.5$$

$$m\overline{LP}=\frac{6\times 2.5}{7.5}$$

$$m\overline{LP}=2\text{cm}$$

4. In the shown figure, let $m\overline{PA}=8x-7$, $m\overline{PB}=4x-3$, $m\overline{AQ}=5x-3$, $m\overline{BR}=3x-1$. Find the value of x if $\overline{AB}\parallel\overline{QR}$.



Solution: Given:

$m\overline{PA}=8x-7, m\overline{PB}=4x-3, m\overline{AQ}=5x-3,$
$m\overline{BR}=3x-1$

To find:

$x=?$

We know that

$$\frac{m\overline{PA}}{m\overline{AQ}}=\frac{m\overline{PB}}{m\overline{BR}}$$

$$\frac{8x-7}{5x-3}=\frac{4x-3}{3x-1}$$

$$(8x-7)(3x-1)=(4x-3)(5x-3)$$

$$24x^2-8x-21x+7=20x^2-12x-15x+9$$

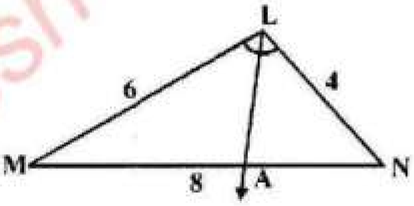
$$24x^2-29x+7=20x^2-27x+9$$

$$24x^2-20x^2-29x+27x+7-9=0$$

$$4x^2-2x-2=0$$
$$2(2x^2-x-1)=0$$
$$\Rightarrow \quad 2x^2-x-1=0$$
$$2x^2-2x+x-1=0$$
$$2x(x-1)+1(x-1)=0$$
$$(x-1)(2x+1)=0$$
$$\Rightarrow \quad x-1=0 \quad \text{or} \quad 2x+1=0$$
$$x=1 \qquad\qquad 2x=-1$$
$$x=\frac{-1}{2}\text{(Not possible)}$$

Hence, $x=1.$

5. In ΔLMN shown in the figure, $\overrightarrow{LA}$ bisects $\angle$L. If $m\overline{LN}=4$, $m\overline{LM}=6, m\overline{MN}=8$, then find $m\overline{MA}$ and $m\overline{AN}$.



Solution: Given:

$$m\overline{LN}=4, m\overline{LM}=6, \quad m\overline{MN}=8\text{cm}$$

To find: $m\overline{MA}=?, m\overline{AN}=?$

$$m\overline{MA}+m\overline{AN}=m\overline{MN}$$
$$m\overline{AN}=m\overline{MN}-m\overline{MA}$$
$$m\overline{AN}=8-m\overline{MA} \qquad \text{\_\_\_\_\_\_\_(1)}$$

We know that $\dfrac{m\overline{AN}}{m\overline{LN}}=\dfrac{m\overline{MA}}{m\overline{LM}}$

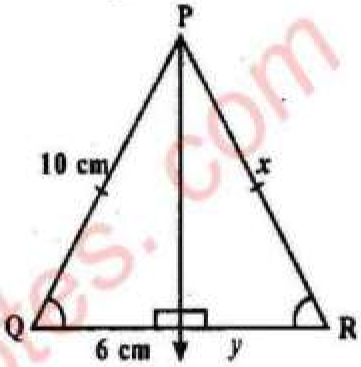
$$\frac{8-m\overline{MA}}{4}=\frac{m\overline{MA}}{6}$$
$$4m\overline{MA}=6\left(8-m\overline{MA}\right)$$
$$4m\overline{MA}=48-6m\overline{MA}$$
$$4m\overline{MA}+6m\overline{MA}=48$$
$$10m\overline{MA}=48$$
$$m\overline{MA}=\frac{48}{10}=\frac{24}{5}\text{cm}$$

Put $m\overline{MA}=\dfrac{24}{5}$ in eq(1), we get

$$m\overline{AN} = 8 - \frac{24}{5} = \frac{40-24}{5} = \frac{16}{5}\text{cm}$$

6. In Isosceles ΔPQR shown in the figure, find the value of x and y.

Solution: As ΔPQR is an isosceles triangle, so



$$m\overline{PR} = m\overline{PQ}$$
$$x = 10\text{cm}$$

And $$m\overline{QS} = m\overline{SR}$$
$$6\text{cm} = y$$
$$\Rightarrow \quad y = 6\text{cm}$$

SUMMARY

In this unit we stated and proved the following theorems and gave some necessary definitions:

- A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.
- If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.
- The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.
- If two triangles are similar, then the measures of their corresponding sides are proportional.
- The ratio between two alike quantities is defined as a: b $=\frac{a}{b}$, where a and b are the elements of the ratio.
- Proportion is defined as the equality of two ratios i.e. a: b = c : d.
- Two triangles are said to be similar if there are equiangular and corresponding sides are proportional.

UNIT 15

PYTHAGORAS' THEOREM

Unit Outlines

★ *Pythagoras' Theorem*

STUDENTS LEARNING OUTCOMES

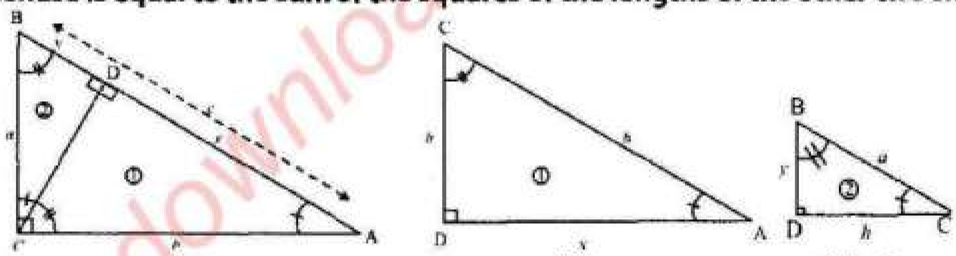
After studying this unit the students will be able to:

- Prove that in a right-angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides. (Pythagoras' theorem).
- Prove that if the square of one side of a triangle is equal to the sum of the squares of the other two sides then the triangle is a right, angled triangle (converse to Pythagoras' theorem).

Introduction:

Pythagoras, a Greek philosopher and mathematician discovered the simple but important relationship between the sides of a right-angled triangle. He formulated this relationship in the form of a theorem called Pythagoras' Theorem.

Pythagoras Theorem 15.1.1: In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.



(i) (ii) a (ii) b

Solution: Given:

ΔACB is a right angled triangle in which $m\angle C = 90^\circ$ and $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$.

To Prove:

$$c^2 = a^2 + b^2$$

Construction:

Draw $\overline{CD}$ perpendicular from C on $\overline{AB}$.

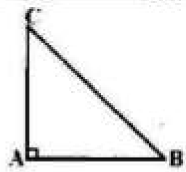
Let $m\overline{CD} = h$, $m\overline{AD} = x$ and $m\overline{BD} = y$. Line segment CD splits ΔABC into two Δs ADC and BDC which are separately shown in the figures (ii) –a and (ii) –b respectively.

Proof (Using similar Δs):

Statements	Reasons
In $\Delta ADC \leftrightarrow \Delta ACB$	Refer to figure (ii)-a and (i)
$\angle A \cong \angle A$	common - self congruent
$\angle ADC \cong \angle ACB$	Construction - given, each angle = 90°
$\angle C \cong \angle B$	$\angle C$ and $\angle B$, complements of $\angle A$.
$\therefore \quad \Delta ADC \sim \Delta ACB$	Congruency of three angles
$\therefore \quad \frac{x}{b} = \frac{b}{c}$	(Measures of corresponding sides of similar triangles are proportional)
Or $\quad x = \frac{b^2}{c}$ .... (i)	
Again in $\Delta BDC \leftrightarrow \Delta BCA$	Refer to figure (ii)-b and (i)
$\angle B \cong \angle B$	Common-self congruent
$\angle BDC \cong \angle BCA$	Construction - given, each angle = 90°
$\angle C \cong \angle A$	$\angle C$ and $\angle A$, complements of $\angle B$
$\therefore \quad \Delta BDC \sim \Delta BCA$	Congruency of three angles.
$\therefore \quad \frac{y}{a} = \frac{a}{c}$	(Corresponding sides of similar triangles are proportional).
or $\quad y = \frac{a^2}{c}$ .... (ii)	
But $\quad y + x = c$	Supposition.
$\therefore \quad \frac{a^2}{c} + \frac{b^2}{c} = c$	By (i) and (ii)
Or $\quad a^2 + b^2 = c^2$	Multiplying both sides by c.
i.e., $\quad c^2 = a^2 + b^2$	

Corollary:

In a right angled ΔABC, right angled at A,



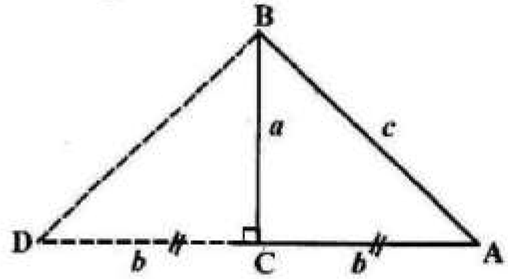
(i) $\overline{AB}^2 = \overline{BC}^2 - \overline{CA}^2$

(ii) $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$

Theorem 15.1.2: [Converse of Pythagoras' Theorem 15.1.1]

If the square of one side of a triangle is equal to the sum of the sum of the other two sides then the triangle is a right angled triangle.

Given: In a ΔABC, $m\overline{AB} = c$, $m\overline{BC} = a$ and $m\overline{AC} = b$. Such that $a^2 + b^2 = c^2$.



To Prove:

ΔACB is a right angled triangle.

Construction:

Draw $\overline{CD}$ perpendicular to $\overline{BC}$ such that $\overline{CD} \cong \overline{CA}$. Join the points B and D.

Proof:

Statements	Reasons
ΔDCB is a right-angled triangle.	Construction
$\therefore \quad \left(m\overline{BD}\right)^2 = a^2 + b^2$	Pythagoras theorem
But $\quad a^2 + b^2 = c^2$	Given
$\therefore \quad \left(m\overline{BD}\right)^2 = c^2$	
or $\quad m\overline{BD} = c$	Taking square root of both sides.
Now in	
$\Delta DCB \leftrightarrow \Delta ACB$	
$\overline{CD} \cong \overline{CA}$	Construction
$\overline{BC} \cong \overline{BC}$	Common
$\overline{DB} \cong \overline{AB}$	Each side = c.
$\therefore \quad \Delta DCB \cong \Delta ACB$	S.S.S. ≅ S.S.S.
$\therefore \quad \angle DCB \cong \angle ACB$	(Corresponding angles of congruent triangles)
But $\quad m\angle DCB = 90^\circ$	Construction
$\therefore \quad m\angle ACB = 90^\circ$	
Hence the ΔACB is a right-angled triangle.	

Corollary:

Let c be the longest of the sides a, b and c of a triangle.

* If $a^2 + b^2 = c^2$, then the triangle is right.
* If $a^2 + b^2 > c^2$, then the triangle is acute.
* If $a^2 + b^2 < c^2$, then the triangle is obtuse.

Solved Exercise 15

1. Verify that the Δ s having the following measures of sides are right-angled.

(i) $a = 5$ cm, $b = 12$ cm, $c = 13$ cm

Solution: $a = 5\text{cm}, \quad b = 12\text{cm}, \quad c = 13\text{cm}$

We know that

$$c^2 = a^2 + b^2$$
$$(13)^2 = (5)^2 + (12)^2$$
$$169 = 25 + 144$$
$$169 = 169$$

Hence given sides represents right angle triangle.

(ii) $a = 1.5$ cm, $b = 2$ cm, $c = 2.5$ cm

Solution: $a = 1.5\text{cm}, \quad b = 2\text{cm}, \quad c = 2.5\text{cm}$

We know that

$$c^2 = a^2 + b^2$$
$$(2.5)^2 = (1.5)^2 + (2)^2$$
$$6.25 = 2.25 + 4$$
$$6.25 = 6.25$$

Hence given sides represents right angle triangle.

(iii) $a = 9$ cm, $b = 12$ cm, $c = 15$ cm

Solution: $a = 9$ cm, $b = 12$ cm, $c = 15$ cm

$$c^2 = a^2 + b^2$$
$$(15)^2 = (9)^2 + (12)^2$$
$$225 = 81 + 144$$
$$225 = 225$$

Hence given sides represents right angle triangle.

(iv) $a = 16$ cm, $b = 30$ cm, $c = 34$ cm

Solution: $a = 16$ cm, $b = 30$ cm, $c = 34$ cm

$$c^2 = a^2 + b^2$$
$$(34)^2 = (16)^2 + (30)^2$$
$$1156 = 256 + 900$$
$$1156 = 1156$$

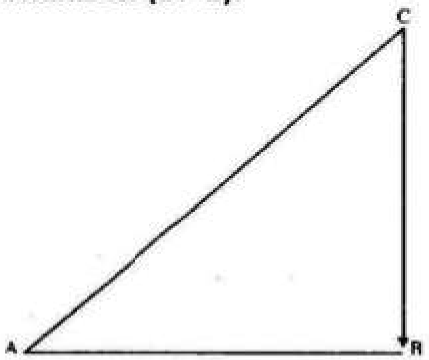
Hence given sides represents right angle triangle.

2. Verify that $a^2 + b^2$, $a^2 - b^2$ and $2ab$ are the measures of the sides of a right angled triangle where a and b are any two real numbers $(a > b)$.

Solution: Given:

In ΔABC

$$\overline{AC} = a^2 + b^2, \overline{AB} = a^2 - b^2, \overline{BC} = 2ab$$



To Prove:

ABC is a right angle triangle.

Proof:

If ABC is a right angle triangle then Pythagoras theorem holds. i.e.,

$$|\overline{AC}|^2 = |\overline{AB}|^2 + |\overline{BC}|^2$$
$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2$$
$$(a^2 + b^2)^2 = a^4 + b^4 - 2a^2b^2 + 4a^2b^2$$
$$(a^2 + b^2)^2 = a^4 + b^4 + 2a^2b^2$$

$$\left(a^2+b^2\right)^2=\left(a^2+b^2\right)^2$$

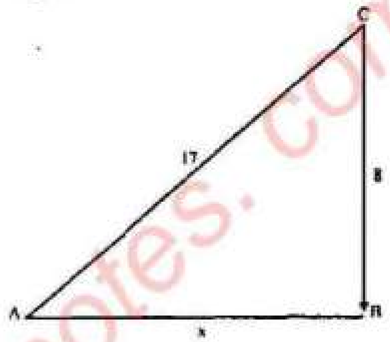
Hence, Pythagoras theorem holds, therefore AB is a right angle triangle.

3. The three sides of a triangle are of measure 8, x and 17 respectively. For what value of x will it become base of a right angled triangle?

Solution: Given:

In ΔABC

$\overline{AB}=x, \quad \overline{BC}=8, \quad \overline{AC}=17$



To Prove:

ABC is a right angle triangle then Pythagoras theorem holds. i.e;

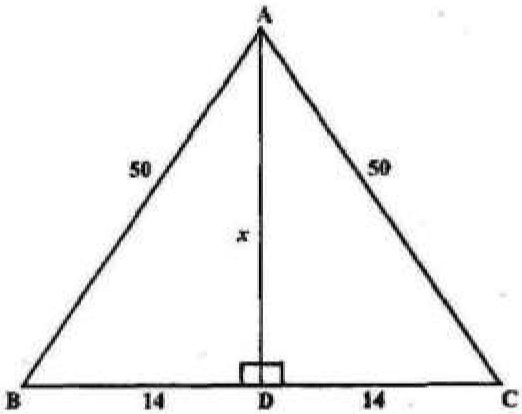
$$|\overline{AC}|^2=|\overline{AB}|^2+|\overline{BC}|^2$$
$$(17)^2=(x)^2+(8)^2$$
$$289=x^2+64$$
$$x^2=289-64$$
$$x^2=225$$
$$\sqrt{x^2}=\sqrt{225}$$
$$x=15$$

4. In an isosceles Δ, the base $m\overline{BC}=28$ cm, and $m\overline{AB}=m\overline{AC}=50$ cm. If $\overline{AD}\perp\overline{BC}$, then find (i) length of $\overline{AD}$ (ii) area of ΔABC

Solution:

Base $\overline{BC}=28\text{cm}, \quad \overline{AB}=\overline{AC}=50\text{cm}$.

and $\overline{AP}\perp\overline{BC}$



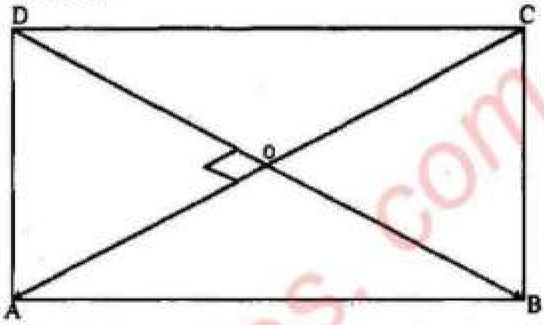
(i) $|\overline{AB}|^2=|\overline{AD}|^2+|\overline{BD}|^2$

$$(50)^2=|\overline{AD}|^2+(14)^2$$
$$|\overline{AD}|^2=(50)^2-(14)^2$$
$$|\overline{AD}|^2=2500-196$$
$$|\overline{AD}|^2=2304$$
$$\overline{AD}=\sqrt{2304}=48\text{cm}$$

(ii) Area of $\Delta ABC=\frac{1}{2}(\text{Base})(\text{Altitude})$

$$=\frac{1}{2}(48)(28)$$
$$=672\text{cm}^2$$

5. In a quadrilateral ABCD, the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other. Prove that $m\overline{AB}^2 + m\overline{CD}^2 = m\overline{AD}^2 + m\overline{BC}^2$.



Solution: Given:

In a quadrilateral ABCD, the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other.

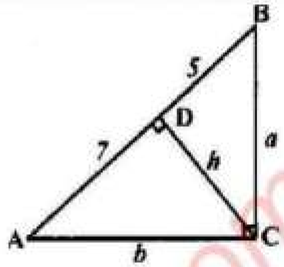
To Prove:

$$m\overline{AB}^2 + m\overline{CD}^2 = m\overline{AD}^2 + m\overline{BC}^2$$

Proof:

Statements	Reasons
In ΔOAD	
$\overline{AD}^2 = \overline{OA}^2 + \overline{OD}^2$ ____ (i)	By Pythagoras Theorem
In ΔOAB	
$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2$ ____ (ii)	By Pythagoras Theorem
In ΔOBC	
$\overline{BC}^2 = \overline{OB}^2 + \overline{OC}^2$ ____ (iii)	By Pythagoras Theorem
In ΔOCD	
$\overline{CD}^2 = \overline{OC}^2 + \overline{OD}^2$ ____ (iv)	By Pythagoras Theorem
$\overline{AD}^2 - \overline{AB}^2 = \left(\overline{OA}^2 + \overline{OD}^2\right) - \left(\overline{OA}^2 + \overline{OB}^2\right)$	Subtract (ii) from (i)
$= \overline{OA}^2 + \overline{OD}^2 - \overline{OA}^2 - \overline{OB}^2$	
$= \overline{OD}^2 - \overline{OB}^2$ ____ (v)	
$\overline{BC}^2 - \overline{CD}^2 = \left(\overline{OB}^2 + \overline{OC}^2\right) - \left(\overline{OC}^2 + \overline{OD}^2\right)$	Subtract (iv) from (ii)
$= \overline{OB}^2 + \overline{OC}^2 - \overline{OC}^2 - \overline{OD}^2$	
$= -\overline{OD}^2 + \overline{OB}^2$	
$= -(\overline{OD}^2 - \overline{OB}^2)$	
$-\left(\overline{BC}^2 - \overline{CD}^2\right) = \overline{OD}^2 - \overline{OB}^2$	
$\overline{CD}^2 - \overline{BC}^2 = \overline{OD}^2 - \overline{OB}^2$ ____ (vi)	Comparing (v) and (vi)
$\overline{AD}^2 - \overline{AB}^2 = \overline{CD}^2 - \overline{BC}^2$	
$\overline{AD}^2 + \overline{BC}^2 = \overline{AB}^2 + \overline{CD}^2$	Equating (v) and (vi)
Or $m\overline{AB}^2 + m\overline{CD}^2 = m\overline{AD}^2 + m\overline{BC}^2$	

6.(i) In the ΔABC as shown in the figure, $m\angle ACB = 90^\circ$ and $\overline{CD} \perp \overline{AB}$. Find the lengths a, h and b if $m\overline{BD} = 5$ units and $m\overline{AD} = 7$ units.



Solution: (i) Given:

$m\angle ACB = 90^\circ$ and $\overline{CD} \perp \overline{AB}$.

$m\overline{BD} = 5,\ m\overline{AD} = 7,\ m\overline{AC} = b,\ m\overline{CD} = h, m\overline{BC} = a$

To find:

$a = ?,\quad h = ?,\quad b = ?$

In ΔBDC

$$(\overline{BC})^2 = (\overline{BD})^2 + (\overline{CD})^2$$

$$(a)^2 = (5)^2 + (h)^2$$

$$a^2 = 25 + h^2 \quad \text{\_\_\_\_\_\_(i)}$$

In ΔADC

$$(\overline{AC})^2 = (\overline{AD})^2 + (\overline{CD})^2$$

$$(b)^2 = (7)^2 + (h)^2$$

$$b^2 = 49 + h^2 \quad \text{\_\_\_\_\_\_(ii)}$$

Subtract eq. (i) from eq. (ii), we get

$$b^2 - a^2 = (49 + h^2) - (25 + h^2)$$

$$b^2 - a^2 = 49 + h^2 - 25 - h^2$$

$$b^2 - a^2 = 24 \quad \text{\_\_\_\_\_\_(iii)}$$

In right angle ΔABC

$$(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2$$

$$(7+5)^2 = (b^2) + (a)^2$$

$$(12)^2 = b^2 + a^2$$

$$b^2 + a^2 = 144 \quad \text{\_\_\_\_\_\_(iv)}$$

By adding eg. (iii) and (iv), we get

$$(b^2 - a^2) + (b^2 + a^2) = 24 + 144$$

$$b^2 - a^2 + b^2 + a^2 = 168$$

$$2b^2 = 168$$

$$b^2 = 84$$

$$b = 2\sqrt{21}$$

Put $b^2 = 84$ in eq. (iv), we get

$$84 + a^2 = 144$$

$a^2 = 144 - 84 = 60$

$a = \sqrt{60} \quad = \sqrt{4\times15} = 2\sqrt{15}$

Put $a^2 = 60$ in eq. (i), we get

$a^2 = 25 + h^2$

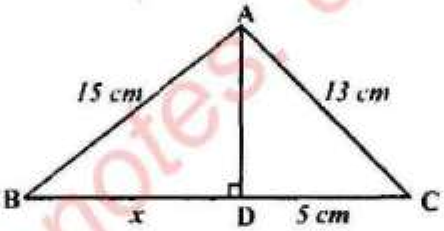
$60 = 25 + h^2$

$h^2 = 60 - 25 = 35$

$h = \sqrt{35}$

So, $a = 12\sqrt{15}$, $b = 2\sqrt{21}$, $h = \sqrt{35}$

(ii) Find the value of x in the shown figure.



Solution: In right angle ΔACD

$$\left(\overline{AC}\right)^2 = \left(\overline{AD}\right)^2 + \left(\overline{CD}\right)^2$$

$$(13)^2 = \left(\overline{AD}\right)^2 + (5)^2$$

$$169 = \left(\overline{AD}\right)^2 + 25$$

or

$$\left(\overline{AD}\right)^2 = 169 - 25$$

$$\left(\overline{AD}\right)^2 = 144$$

$$\sqrt{(AD)^2} = \sqrt{144}$$

$$\overline{AD} = 12$$

In right angle ΔABD.

$$\left(\overline{AB}\right)^2 = \left(\overline{AD}\right)^2 + \left(\overline{BD}\right)^2$$

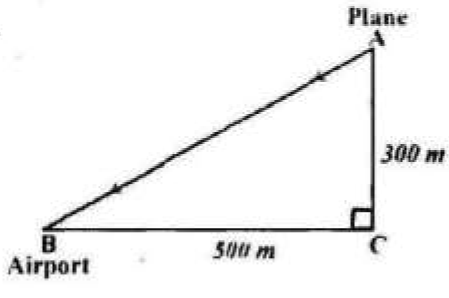
$$(15)^2 = (12)^2 + (x)^2$$

$$x^2 = (15)^2 - (12)^2$$

$$x^2 = 225 - 144$$

$$x^2 = 81 \Rightarrow \sqrt{x^2} = \sqrt{81} \Rightarrow x = 9\,\text{cm}$$

7. A plane is at a height of 300 m and is 500 m away from the airport as shown in the figure. How much distance will it travel to land at the airport?



Solution: By Pythagoras theorem, we have

$$\left(\overline{AB}\right)^2 = \left(\overline{AC}\right)^2 + \left(\overline{BC}\right)^2$$

$$= (300)^2 + (500)^2$$

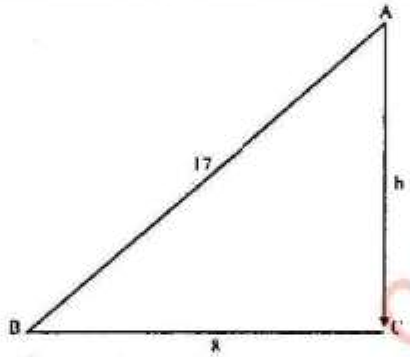
$$= 90000 + 250000 = 340000$$

$$\overline{AB} = \sqrt{34\times10000} = 100\sqrt{34}\text{m}$$

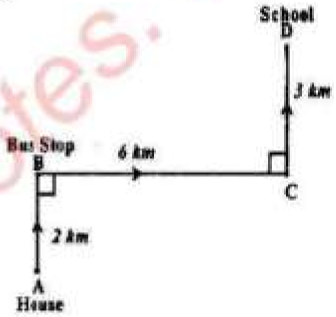
8. A ladder 17 m long rests against a vertical wall. The foot of the ladder is 8 m away from the base of the wall. How high up the wall will the ladder reach?

Solution: By Pythagoras theorem, we have

$$\left(\overline{AB}\right)^2 = \left(\overline{AC}\right)^2 + \left(\overline{BC}\right)^2$$
$$(17)^2 = (h)^2 + (8)^2$$
$$289 = h^2 + 64$$
$$h^2 = 289 - 64 = 225$$
$$h = \sqrt{225}$$
$$h = 15m$$



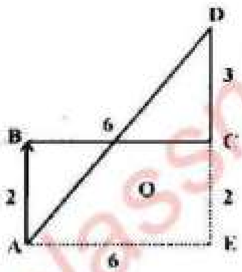
9. A student travels to his school by the route as shown in the figure. Find $m\overline{AD}$, the direct distance from his house to school.



Solution:

In right angle ΔAED

$$\left(\overline{AD}\right)^2 = \left(\overline{AE}\right)^2 + \left(\overline{DE}\right)^2$$
$$= (6)^2 + (3+2)^2$$
$$= (6)^2 + (5)^2 = 36 + 25 = 61$$
$$\overline{AD} = \sqrt{61} \text{ km}$$



Solved Review Exercise 15

1. Which of the following are true and which are false?

(i) In a right angled triangle greater angle is of 90°. ________

(ii) In a right angled triangle right angle is of 60°. ________

(iii) In a right triangle hypotenuse is a side opposite to right angle. ________

(iv) If a, b, c are sides of right angled triangle with c as longer side then $c^2 = a^2 + b^2$ ________

(v) If 3 cm and 4 cm are two sides of a right angled triangle, then hypotenuse is 5cm. ________

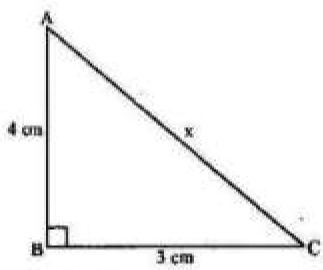
(vi) If hypotenuse of an isosceles right triangle is $\sqrt{2}$ cm, then each of other side is of length 2 cm. ________

Solution: (i) T (ii) F (iii) T (iv) T (v) T (vi) F

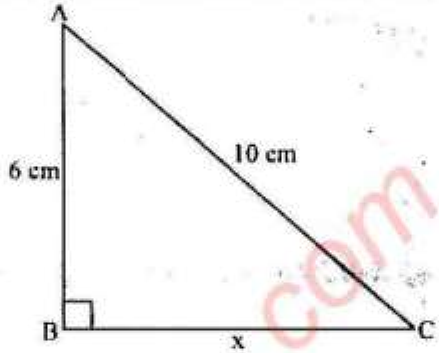
2. Find the unknown value in each of the following figures.

(i) **Solution:** By Pythagoras theorem, we have

$$\left(\overline{AC}\right)^2 = \left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2$$
$$x^2 = (4)^2 + (3)^2$$
$$x^2 = 16 + 9 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = 5cm$$



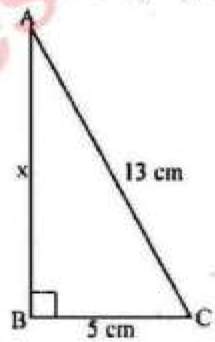
(ii) Solution:



By Pythagoras theorem, we have

$$\left(\overline{AC}\right)^2 = \left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2$$
$$(10)^2 = (6)^2 + (x)^2$$
$$100 = 36 + x^2$$
$$x^2 = 100 - 36 = 64$$
$$\sqrt{x^2} = \sqrt{64}$$
$$x = 8\text{cm}$$

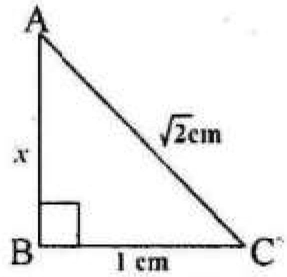
(iii) Solution:



By Pythagoras theorem, we have

$$\left(\overline{AC}\right)^2 = \left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2$$
$$(13)^2 = (x)^2 + (5)^2$$
$$169 = x^2 + 25$$
$$x^2 = 169 - 25 = 144$$
$$\sqrt{x^2} = \sqrt{144}$$
$$x = 12\text{cm}$$

(iv) Solution:



By Pythagoras theorem, we have

$$\left(\overline{AC}\right)^2 = \left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2$$
$$\left(\sqrt{2}\right)^2 = (x)^2 + (1)^2$$
$$2 = x^2 + 1$$
$$x^2 = 2 - 1 = 1$$
$$\sqrt{x^2} = \sqrt{1}$$
$$x = 1\text{cm}$$

SUMMARY

In this unit we learned to state and prove Pythagoras' Theorem and its converse with corollaries.

* In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.
* If the square of one side of a triangle is equal to the sum of the squares of the other two sides then the triangle is a right angled triangle.

Moreover, these theorems were applied to solve some questions of practical use.

UNIT 16

THEOREMS RELATED WITH AREA

Unit Outlines
★ Theorems related with area

STUDENTS LEARNING OUTCOMES

After studying this unit the students will be able to:

- Prove that parallelograms on the same base and lying between the- same parallel lines (or of the same altitude) are equal in area.
- Prove that parallelograms on equal bases and having the same altitude are equal in area.
- Prove that triangles on the same base and of the same altitude are equal in area.
- Prove that triangles on equal bases and of the same altitude are equal in area.

Introduction:

In this unit we will state and prove some important theorems related with area of parallelograms and triangles along with corollaries. We shall apply them to solve appropriate problems and to prove some useful results.

Some Preliminaries:

Area of a Figure:

The region enclosed by the bounding lines of a closed figure is called the **area of the figure.**

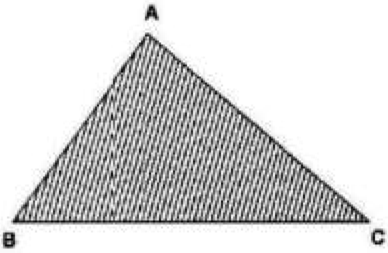
The area of a closed region is expressed in square units (say, sq. m or m^2) i.e., a positive real number.

Triangular Region:

The interior of a triangle is the part of the plane enclosed by the triangle.

A **triangular region** is the union of a triangle and its interior i.e., the three line segments forming the triangle and its interior.

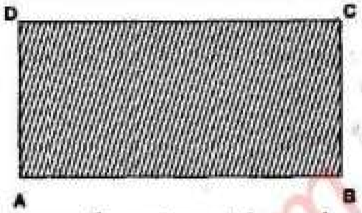
By area of a triangle, we mean the area of its triangular region.



Congruent Area Axiom:

If $\Delta ABC \cong \Delta PQR$, then area of (region ΔABC) = area of (region ΔPQR.)

Rectangular Region:



The interior of a rectangle is the part of the plane enclosed by the rectangle.

A rectangular region is the union of a rectangle and its interior.

A rectangular region can be divided into two or more than two triangular regions in many ways.

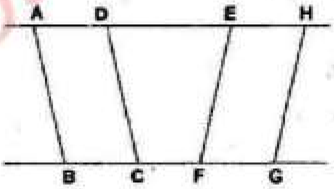
Area of a Rectangle:

If the length and width of a rectangle are 'a' units and 'b' units respectively, then the area of the rectangle is equal to "a × b" square units.

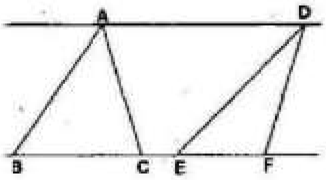
Remember:

If 'a' is the side of a square, its area = a^2, square units.

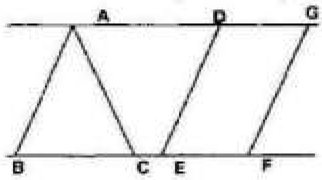
Between the same Parallels:



Two parallelograms are said to be between the same parallels, when their bases are in the same straight line and their sides opposite to these bases are also in a straight line; as the parallelograms ABCD, EFGH in the given figure.



Two triangles are said to be between the same parallels, when their bases are in the same straight line and the line joining their vertices is parallel to their bases; as the Δs ABC, DEF in the given figure.



A triangle and a parallelogram are said to be between the same parallels, when their bases are in the same straight line, and the side of the parallelogram opposite the base, produced if necessary, passes through the vertex of the triangle as are the ΔABC and the parallelogram DEFG in the given figure.

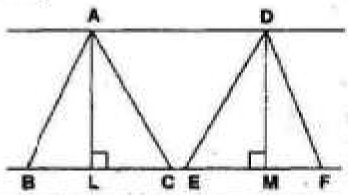
Altitude or Height of the parallelogram:

If one side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it, is called the Altitude or Height of the parallelogram.

Altitude or Height of the triangle:

If one side of a triangle is taken as its base the perpendicular to that side, from the opposite vertex is called the Altitude or Height of the triangle.

Useful Result:



"Triangles or parallelograms having the same or equal altitudes can be placed between the same parallels, and conversely".

Place the triangles ABC, DEF so that their bases

$\overline{BC}$, $\overline{EF}$ are in the same straight line and the vertices on the same side of it, and suppose $\overline{AL}$, $\overline{DM}$ are the equal altitudes. We have to show that $\overline{AD}$ is parallel to BCEF.

Proof:

$\overline{AL}$ and $\overline{DM}$ are parallel, for they are both perpendicular to $\overline{BF}$. Also $m\overline{AL} = m\overline{DM}$. (given)

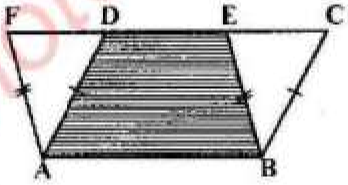
$\therefore$ $\overline{AD}$ is parallel to $\overline{LM}$.

Useful Result:

A diagonal of a parallelogram divides it into two congruent triangles (S.S.S.) and hence of equal area.

Theorem 16.1.1: Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.

Solution: Given:



Two parallelograms ABCD and ABEF having the same base $\overline{AB}$ and between the same parallel lines $\overline{AB}$ and $\overline{DE}$.

To Prove: Area of parallelogram ABCD = Area of parallelogram ABEF

Proof:

Statements	Reasons
Area of (parallelogram ABCD) =Area of (quad. ABED) + area of (ΔCBE)…(1)	[Area addition axiom]
Area of (parallelogram ABEF) =Area of (quad. ABED) + area of (ΔDAF)…..(2)	[Area addition axiom]
In Δs CBE and DAF	
$m\overline{CB} = m\overline{DA}$	[Opposite sides of a parallelogram]
$m\overline{BE} = m\overline{AF}$	[Opposite sides of a parallelogram]
$\angle CBE \cong \angle DAF$	[$\therefore$ BC $\parallel$ AD, BE $\parallel$ AF]
$\therefore$ $\Delta CBE \cong \Delta DAF$	[S.A.S. cong. Axiom]
$\therefore$ Area of (ΔCBE) = Area of (ΔDAF)……(3)	[cong. area axiom]
Hence area of (parallelogram ABCD) = Area of (parallelogram ABEF)	from (1), (2) and (3)

Corollary:

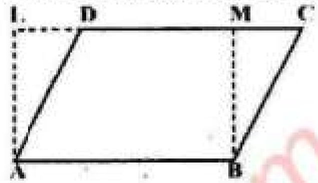
(i) The area of a parallelogram is equal to that of a rectangle on the same base and having the same altitude.

(ii) Hence area of parallelogram = base × altitude

Proof:

Let ABCD be a parallelogram. AL is an altitude corresponding to side AB.

(i) Since parallelogram ABCD and rectangle ALMB are on the same base AB and between the same parallels,



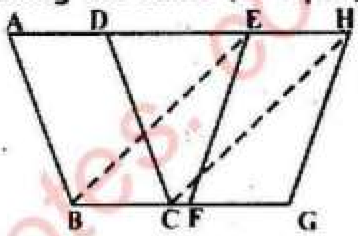
$\therefore$ By above theorem it follows that

Area of (parallelogram ABCD)=Area of (rect. ALMB)

(ii) But area of (rect. ALMB) = AB × AL

Hence area of (parallelogram ABCD) = AB × AL

Theorem 16.1.2: Parallelograms on equal bases and having the same (or equal) altitude are equal in area.



Solution: Given:

Parallelograms ABCD, EFGH are on the equal bases $\overline{BC}$, $\overline{FG}$, having equal altitudes.

To Prove: Area of (parallelogram ABCD) = Area of (parallelogram EFGH)

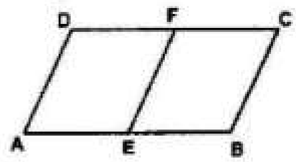
Construction: Place the parallelograms ABCD and EFGH so that their equal bases $\overline{BC}$, $\overline{FG}$ are in the straight line BCFG. Join $\overline{BE}$ and $\overline{CH}$.

Proof:

Statements	Reasons
The given $\|^{gms}$ ABCD and EFGH are between the same parallels	Their altitudes are equal (given)
Hence ADEH is a straight line, which is $\|$ to $\overline{BC}$.	
$\therefore$ $m\overline{BC} = m\overline{FG}$	Given
$= m\overline{EH}$	EFGH is a parallelogram
Now $m\overline{BC} = m\overline{EH}$ and they are $\|$	
$\therefore$ $m\overline{BE}$ and $\overline{CH}$ are both equal and $\|$	
Hence EBCH is a parallelogram	A quadrilateral with two opposite sides congruent and parallel is a parallelogram
Now area of $\|^{gm}$ ABCD = area of $\|^{gm}$ EBCH....(i)	Being on the same base $\overline{BC}$ and between the same parallels
But area of $\|^{gm}$ EBCH = area of $\|^{gm}$ EFGH ...(ii)	Being on the same base $\overline{EH}$ and between the same parallels
Hence area ($\|^{gm}$ ABCD) = area ($\|^{gm}$ EFGH)	From (i) and (ii)

Solved Exercise 16.1

1. Show that the line segment joining the mid-points of opposite sides of a parallelogram, divides it into two equal parallelograms.



Solution: Given:

ABCD is a parallelogram. E is the mid point of side $\overline{AB}$ and F is the mid point of side $\overline{DC}$.

To Prove:

Area of parallelogram AEFD = Area of parallelogram BEFC

Proof: $\overline{AB} \parallel \overline{DC}$ (Opposite sides of parallelogram ABCD)

As E is the mid point of the side $\overline{AB}$. So

$$\overline{AE} \cong \overline{EB}$$

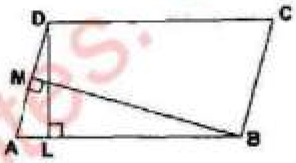
The parallelogram AEFD and BEFC are on equal bases ($\overline{AE} \cong \overline{EB}$) and between the same parallel lines $\overline{AB}$ and $\overline{DC}$.

∴ They have equal areas.

Hence Area of parallelogram AEFD = Area of parallelogram BEFC

2. **In a parallelogram ABCD, m $\overline{AB}$ = 10 cm. The altitudes corresponding to sides $\overline{AB}$ and $\overline{AD}$ are respectively 7 cm and 8 cm. Find $\overline{AD}$.**

Solution: Given:



ABCD is a parallelogram.

m $\overline{AB}$ = 10 cm, $\overline{DL}$ AND $\overline{BM}$ are altitudes

m $\overline{DL}$ = 7 cm, m $\overline{BM}$ = b cm

To Find: m $\overline{AD}$ = ?

Construction: Place the parallelograms so that the bases $\overline{BC}$, $\overline{FG}$ are in the straight line BCFO. Join $\overline{BE}$ and $\overline{CH}$.

Proof: Area of parallelogram = base × altitude.

So Area of parallelogram ABCD = Area of parallelogram ADCB

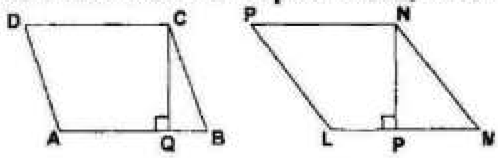
$$m\overline{AB} \times m\overline{DL} = m\overline{AD} \times m\overline{BM}$$

$$10 \times 7 = m\overline{AD} \times 8$$

$$m\overline{AD} = (10 \times 7)/8 = \frac{\cancel{70}^{35}}{\cancel{8}_{4}} = \frac{35}{4}\text{cm}$$

3. **If two parallelograms of equal areas have the same or equal bases, their altitudes are equal.**

Solution: Given:



In parallelogram ABCD, $\overline{CQ}$ is the altitude and in parallelogram LMNP, $\overline{NP}$ the altitude. Area of parallelogram ABCD = Area of parallelogram LMNP and m $\overline{AB}$ = m $\overline{LM}$.

To Prove:

$$m\overline{CQ} = m\overline{NP}$$

Proof:

Area of parallelogram ABCD = Area of parallelogram LMNP (Given)

We know that

Area of parallelogram = base × altitude

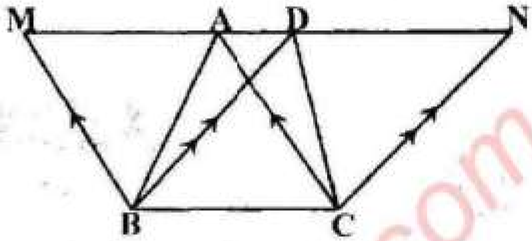
Area of parallelogram ABCD = Area of parallelogram LMNP

$$m\overline{AB} \times m\overline{CQ} = m\overline{LM} \times m\overline{NP}$$

But $$m\overline{AB} = m\overline{LM} \text{ (Given)}$$

$$m\overline{CQ} = m\,NP$$

Theorem 16.1.3: Triangles on the same base and of the same (i.e., equal) altitudes are equal in area.



Solution: Given:

Δs ABC, DBC on the same base, $\overline{BC}$, and having equal altitudes.

To Prove

Area of (ΔABC) = Area of (ΔDBC)

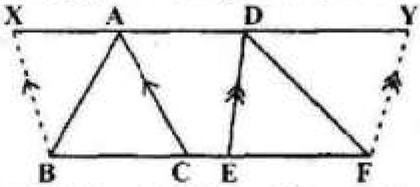
Construction:

Draw $\overline{BM} \parallel$ to $\overline{CA}$, $\overline{CN} \parallel$ to $\overline{BD}$ meeting $\overline{AD}$ produced in M, N.

Proof:

Statements	Reasons
ΔABC and ΔDBC are between the same $\parallel^s$ Hence MADN is parallel to $\overline{BC}$	Their altitudes are equal
$\therefore$ Area ($\parallel^{gm}$ BCAM) = Area ($\parallel^{gm}$ BCND) …(i)	These $\parallel^{gms}$ are on the same base $\overline{BC}$ and between the same $\parallel^s$
But area of ΔABC = $\frac{1}{2}$ (area of $\parallel^{gm}$ BCAM) ...(ii)	Each diagonal of a $\parallel^{gm}$ bisects it into two congruent triangles
and area of ΔDBC = $\frac{1}{2}$ (area of $\parallel^{gm}$ BCND) ...(iii)	
Hence Area (ΔABC) = Area (ΔDBC)	From (i), (ii) and (iii)

Theorem 16.1.4: Triangles on equal bases and of equal altitudes are equal in area.



Given: Δs ABC, DEF on equal bases $\overline{BC}$, $\overline{EF}$ and having altitudes equal.

To Prove: Area (ΔABC) = Area (ΔDEF)

Construction:

Place the Δs ABC and DEF so that their equal bases $\overline{BC}$ and $\overline{EF}$ are in the same straight line BCEF and their vertices on the same side of it. Draw $\overline{BX} \parallel$ to $\overline{CA}$ and $\overline{FY} \parallel$ to $\overline{ED}$ meeting $\overline{AD}$ produced in X, Y respectively.

Proof:

Statements	Reasons
ΔABC, ΔDEF are between the same parallels $\therefore$ XADY is $\parallel$ to BCEF	Their altitudes are equal (given)
$\therefore$ area ($\parallel^{gm}$ BCAX) = area ($\parallel^{gm}$ EFYD) …….(i)	These $\parallel^{gms}$ are on equal bases and between the same parallels
But area of ΔABC = $\frac{1}{2}$ (area of $\parallel^{gm}$ BCAX) …(ii)	Diagonal of a $\parallel^{gm}$ bisects it
and area of ΔDEF = $\frac{1}{2}$ (area of $\parallel^{gm}$ EFYD) ....(iii)	
$\therefore$ area (ΔABC) = area (ΔDEF)	From (i), (ii) and (iii)

Corollaries:

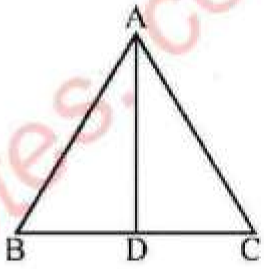
1. Triangles on equal bases and between the same parallels are equal in area.
2. Triangles having a common vertex and equal bases in the same straight line, are equal in area.

Solved Exercise 16.2

1. Show that a median of a triangle divides it into two triangles of equal area.

Solution:

Let ABC be a triangle and D is the mid point of the side $\overline{BC}$ where $\overline{BC} = 2b$



In ΔACD,

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{altitude}$$

$$= \frac{1}{2}\, b \times h = \frac{1}{2} bh$$

In ΔBCD,

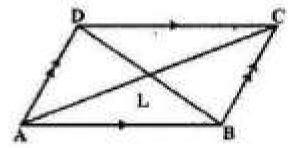
$$\text{Area} = \frac{1}{2}\ \text{base} \times \text{altitude}$$

$$= \frac{1}{2}\, b \times h = \frac{1}{2} bh$$

Since we have two triangle having equal areas, therefore median of a triangle divides it into two triangles of equal areas.

2. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.

Solution: Given:



In parallelogram ABCD, $\overline{AC}$ and $\overline{BD}$ are its diagonal, which meet at L.

To Prove: Triangles ABL, BCL, CDL, and ADL have equal area.

Proof: Triangles ABC and ABD have the same base $\overline{AB}$ and are between the same parallel lines $\overline{AB}$ and $\overline{DC}$.

$\therefore$ They have same area.

Or Area of ΔABC = Area of ΔABD

Area of ΔABL + Area of ΔBCL = Area of ΔABL + Area of ΔADL

Area of ΔBCL = Area of ΔADL ....(i)

Similarly Area of ΔABC = Area of ΔABD

$\Rightarrow$ Area of ΔBCL + Area of ΔABL = Area of ΔBCL + Area of ΔCDL

$\Rightarrow$ Area of ΔABL = Area of ΔCDL ....(ii)

As diagonals of a parallelogram bisect each other.

But L is the mid point of $\overline{AC}$.

So $\overline{BL}$ is a median of $\overline{BC}$.

Area of ΔABL = Area of ΔBCL ....(iii)

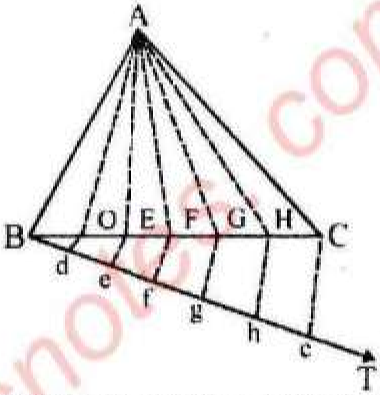
From (i), (ii), (iii), we get
Area of ΔABL = Area of ΔBCL = Area of ΔCDL = Area of ΔADL
Hence Proved

3. Divide a triangle into six equal triangular parts.

Solution: Given: ΔABC

Required: To divide ΔABC into six equal triangular parts.

Construction:



(i) Draw the ray $\overline{BT}$ making an acute $\angle CBT$.
(ii) On $\overline{BT}$ mark six points d, e, f, g, h and c. Such that
$m\overline{Bd} = m\overline{de} = m\overline{ef} = m\overline{fg} = m\overline{gh} = m\overline{hc}$.
(iii) Join c with C.
(iv) Draw $\overline{Hh}$, $\overline{Gg}$, $\overline{Ff}$, $\overline{Ee}$ and $\overline{Od}$ each parallel to $\overline{cC}$.
(v) Join A to O, E, F, G and H. So BAO, OAE, EAF, FAG, GAH and HAC are required six equal triangular parts.

Solved Review Exercise 16

1. Which of the following are true and which are false?
(i) Area of a figure means region enclosed by bounding lines of closed figure.
(ii) Similar figures have same area.
(iii) Congruent figures have same area.
(iv) A diagonal of a parallelogram divides it into two non-congruent triangles.
(v) Altitude of a triangle means perpendicular from vertex to the opposite side (base).
(vi) Area of a parallelogram is equal to the product of base and height.

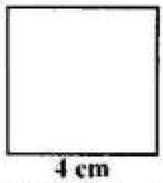
Solution: (i) T (ii) F (iii) T (iv) F (V) T (vi) T

2. Find the area of the following.

(i)

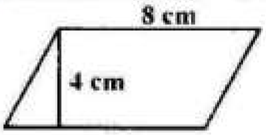
Solution: Area = $6 \times 3 = 18 \text{ cm}^2$

(ii)



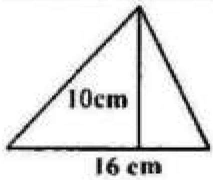
Solution: Area = $4 \times 4 = 16 \text{ cm}^2$

(iii)



Solution: Area = $8 \times 4 = 32 \text{ cm}^2$

(iv)



Solution: Area=$\frac{1}{2}\times16\times10=80 \text{ cm}^2$

3. Define the following

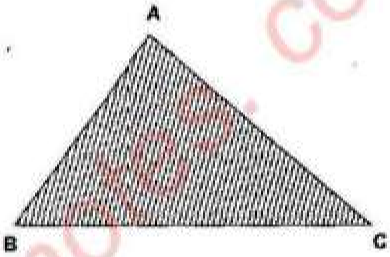
(i) Area of a figure **(ii) Triangular Region**

(iii) Rectangular Region **(iv) Altitude or Height of a triangle**

Solution: (i) Area of a figure:

The region enclosed by the bounding lines of a closed figure is called the **Area of the figure.**

The area of a closed region is expressed in square units (say, sq. m or m^2) i.e. a positive real number.



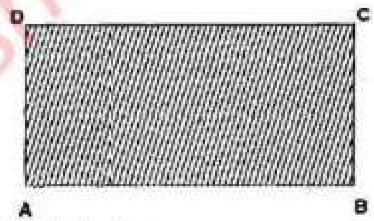
(ii) Triangular Region:

The interior of a triangle is the part of the plane enclosed by the triangle.

A **triangular region** is the union of a triangle and its interior i.e., the three line segments forming the triangle and its interior.

Congruent Area Axiom:

If $\Delta ABC \cong \Delta PQR$, then area of (region ΔABC) = area of (region ΔPQR.)



(iii) Rectangular Region:

The interior of a rectangle is the part of the plane enclosed by the rectangle.

A rectangular region is the union of a rectangle and its interior.

(iv) Altitude or Height of the triangle: If one side of a triangle is taken as its base, the perpendicular distance between that side and the side parallel to it, is called the Altitude or Height of the parallelogram.

SUMMARY

In this unit we mentioned some necessary preliminaries, stated and proved the following theorems along with corollaries, if any.

- Area of a figure means region enclosed by the boundary lines of a closed figure.
- A triangular region means the union of triangle and its interior.
- By area of triangle means the area of its triangular region.
- Altitude or height of a triangle means perpendicular distance to base from its opposite vertex.
- Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.
- Parallelograms on equal bases and having the same (or equal) altitude are equal in area.
- Triangles on the same base and of the same (i.e., equal) altitudes are equal in area,
- Triangles on equal bases and of equal altitudes are equal in area.

UNIT 17

PRACTICAL GEOMETRY TRIANGLES

Unit Outlines

- *Construction of Triangles*
- *Figures with equal areas*

STUDENTS LEARNING OUTCOMES

After studying this unit the students will be able to:

- Construct a triangle having given: two sides and the included angle, one side and two of the angles, two of its sides and the angle opposite to one of them (with all the three possibilities)
- Draw: angle bisectors, attitudes, perpendicular bisectors, medians, of a given triangle and verify their concurrency.
- Construct a triangle equal in area to a given quadrilateral. Construct a rectangle equal in area to a given triangle. Construct a square equal in area to a given rectangle. Construct a triangle of equivalent area on a base of given length.

Construction of Triangle:

(a) To construct a triangle, having given two sides and the included angle:

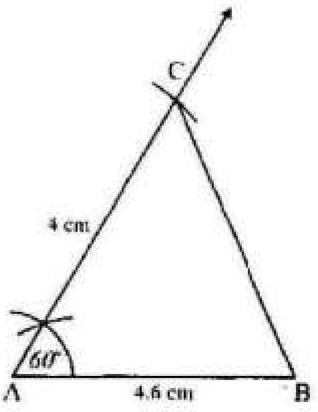
Given: Two sides, say

$m\overline{AB} = 4.6$ cm and $m\overline{AC} = 4$ cm

and the included angle, $m\angle A = 60°$.

Required:

To construct the ΔABC using given information of sides and the included angle $= \angle 60°$



Construction:

(i) Draw a line segment $m\overline{AB} = 4.6$ cm.

(ii) At the end A of $\overline{AB}$ make $m\angle BAC = \angle 60°$.

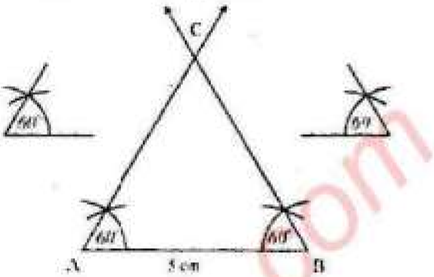
(iii) Cut off $m\overline{AC} = 4$ cm from the terminal side of $\angle 60°$.

(iv) Join B with C.

(v) Then ABC is the required Δ.

(b) To construct a triangle, having given one side and two of the angles:

Given: The side $m\overline{AB} = 5$ cm, say and two of the angles, say $m\angle A = 60^\circ$ and $m\angle B = 60^\circ$.



Required: To construct a ΔABC using given data.

Construction:

(i) Draw the line segment $m\overline{AB} = 5$ cm

(ii) At the end point A of $\overline{AB}$ make $m\angle BAC = \angle 60^\circ$.

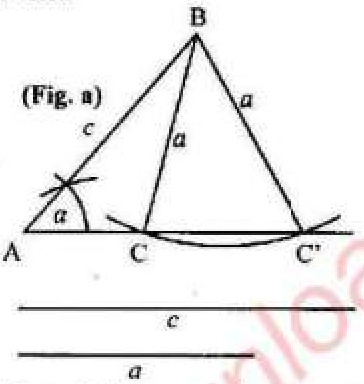
(iii) At the end point B of $\overline{BA}$ make $m\angle ABC = \angle 60^\circ$.

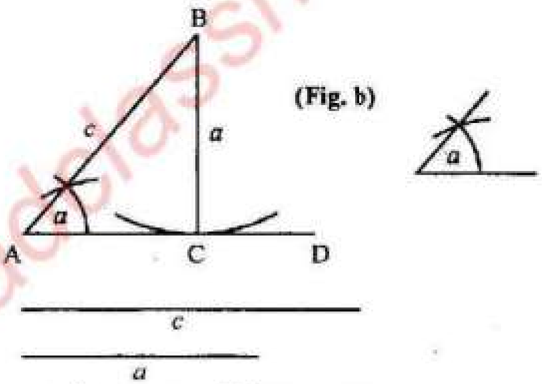
(iv) The terminal sides of these two angles meet at C.

(v) Then ABC is the required Δ.

(c) Ambiguous Case:

To construct a triangle having given two of its sides and the angle opposite to one of them:





Given: Two sides a, c and $\angle A = \propto$ opposite to one of them, say a.

Required: To construct a triangle having the given parts.

Construction:

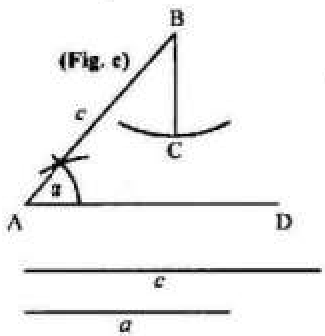
(i) Draw a straight line AD of any length.

(ii) At A make $\angle DAB = \angle A = \propto$

(iii) Cut off $\overline{AB} = c$.

(iv) With centre B and radius equal to a, drawn an arc.

(v) Three cases arise,



Case I: When the arc with radius a cuts $\overline{AD}$ in two distinct points C and C' as in Figure (a).

Joint BC and BC '. Then both the triangles ABC and ABC ' have the given parts and are the required triangles.

Case II: When the arc with radius a only touches $\overline{AD}$ at C, as in Figure (b).

Join BC. Then AABC is the required triangle right

angled at C.

Case III: When the arc with radius a neither cuts nor touches $\overline{AD}$ as in Figure (c). There will be no triangle in this case.

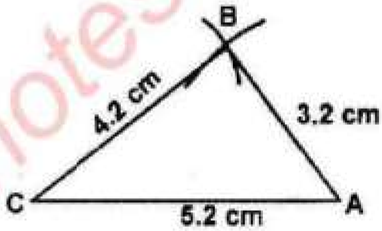
Note:

Recall that in a ΔABC the length of the side opposite to $\angle A$ is denoted by a, opposite to $\angle B$ is denoted by b and opposite to $\angle C$ is denoted by c.

Solved Exercise 17.1

1. Construct a ΔABC, in which

(i) $m\overline{AB} = 3.2cm$, $m\overline{BC} = 4.2$ cm, $m\overline{CA} = 5.2$ cm,



Solution: Given:

$m\overline{AB} = 3.2cm$, $m\overline{BC} = 4.2$ cm, $m\overline{CA} = 5.2$ cm,

Required: To construct the ΔABC using given information of sides.

Construction:

(i) Draw a line segment $m\overline{AC} = 5.2$ cm

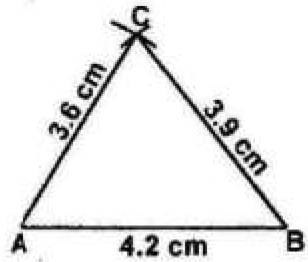
(ii) With center at A, draw an arc of radius 3.2 cm.

(iii) With center at C, draw an arc of radius 4.2 cm which cuts the previous arc at point B.

(iv) Join B with A and C.

(v) Thus ABC is the required triangle.

(ii) $m\overline{AB} = 4.2cm$, $m\overline{BC} = 3.9$ cm, $m\overline{CA} = 3.6$ cm,



Solution: Given:

$m\overline{AB} = 4.2cm$, $m\overline{BC} = 3.9$ cm, $m\overline{CA} = 3.6$ cm,

Required: To construct the ΔABC using given information of sides.

Construction:

(i) Draw a line segment $m\overline{AB} = 4.2$ cm

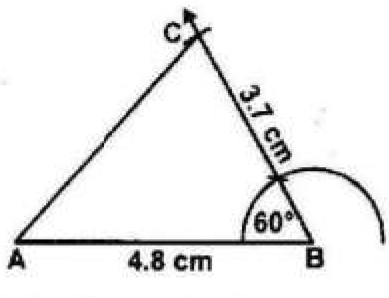
(ii) With center at A, draw an arc of radius 3.6 cm.

(iii) With center at B, draw an arc of radius 3.9 cm which cuts the previous arc at point C.

(iv) Join C with A and B.

(v) Thus ABC is the required triangle.

(iii) $m\overline{AB} = 4.8cm$, $m\overline{BC} = 3.7$ cm, $m\angle B = 60°$



Solution: Given:

$m\overline{AB} = 4.8cm$, $m\overline{BC} = 3.7$ cm, $m\angle B = 60°$

Required: To construct the ΔABC by using given information.

Construction:

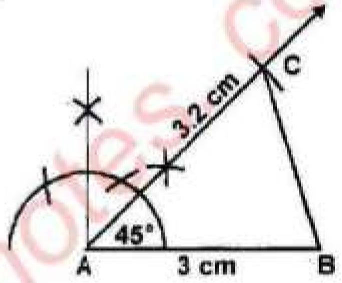
(i) Draw a line segment m $\overline{AB}$ = 4.8 cm
(ii) At the end B of $\overline{AB}$ make m∠B = 60°.
(iii) Cut off m$\overline{BC}$ = 3.7 cm from the terminal side of ∠ABC = 60°.
(iv) Join A with C.
(v) Thus ABC is the required triangle.

(iv) m$\overline{AB}$ = 3cm, m$\overline{AC}$ = 3.2 cm, m∠A = 45°

Solution: Given:

m$\overline{AB}$ = 3cm, m$\overline{AC}$ = 3.2 cm, angle, m∠A = 45°.

Required: To construct the ΔABC by using given information.



Construction:

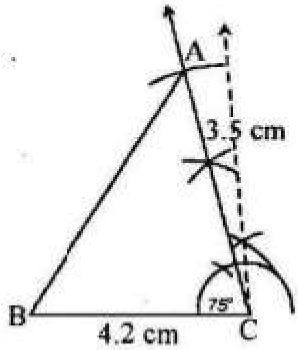
(i) Draw a line segment m $\overline{AB}$ = 3 cm
(i) At the end A of $\overline{AB}$ make m∠A= 45°.
(ii) Cut off m$\overline{AC}$ = 3.2 cm from the terminal side of ∠BAC = 45°.
(iii) Join B with C.
(iv) Thus ABC is the required triangle.

(v) m$\overline{BC}$ = 4.2cm, m$\overline{CA}$ = 3.5 cm, m∠C = 75°

Solution: Given:

m$\overline{BC}$ = 4.2cm, m$\overline{CA}$ = 3.5 cm, m∠C = 75°.

Required: To construct the ΔABC by using given information.



Construction:

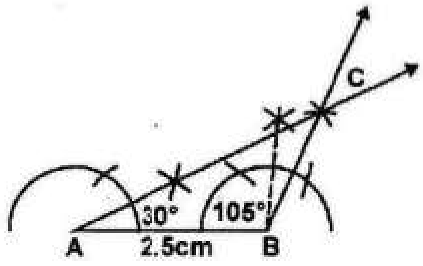
(i) Draw a line segment m$\overline{BC}$ = 4.2 cm
(ii) At the end C of $\overline{BC}$ make m∠C = 75°.
(iii) Cut off m$\overline{CA}$ = 3.5 cm from the terminal side of ∠BCA = 75°.
(iv) Join B with C.
(v) Thus ABC is the required triangle.

(vi) m$\overline{AB}$ = 2.5cm, m∠A = 30°, m∠B = 105°

Solution: Given:

m$\overline{AB}$ = 2.5 cm, m∠A = 30° and m∠B = 105°.

Required: To construct the ΔABC by using given information.



Construction:

(i) Draw a line segment m $\overline{AB}$ = 2.5 cm
(ii) At the end point A of $\overline{AB}$ make m∠BAC = 30°.

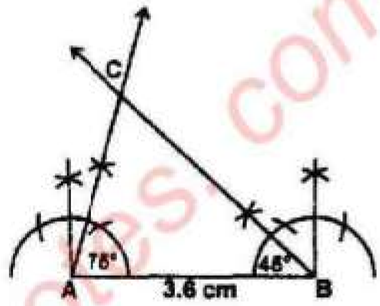
(iii) At the end point B of $\overline{BA}$ make $m\angle ABC = 105^\circ$.
(iv) The terminal sides of these two angles meet at C.
(v) Thus ABC is the required triangle.

(vii) $m\overline{AB} = 3.6cm, m\angle A = 75^\circ, m\angle B = 45^\circ$

Solution: Given:

$m\overline{AB} = 3.6cm$, $m\angle A = 75^\circ$ and $m\ \angle B = 45^\circ$.

Required: To construct the ΔABC by using given information.



Construction:

(i) Draw a line segment $m\overline{AB} = 3.6$ cm
(ii) At the end point A of $\overline{AB}$ make $m\angle BAC = 75^\circ$.
(iii) At the end point B of $\overline{BA}$ make $m\angle ABC = 45^\circ$.
(iv) The terminal sides of these two angles meet at C.
(v) Thus ABC is the required triangle.

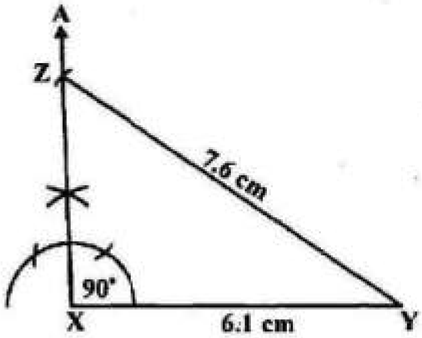
2. Construct a ΔXYZ, in which

(i) $m\overline{YZ} = 7.6$ cm, $m\overline{XY} = 6.1$cm and $m\angle X = 90^\circ$

Solution: Given:

$m\overline{YZ} = 7.6cm$, $m\overline{XY} = 6.1$ cm, $m\angle X = 90^\circ$

Required: To construct the ΔXYZ by using given information.



Construction:

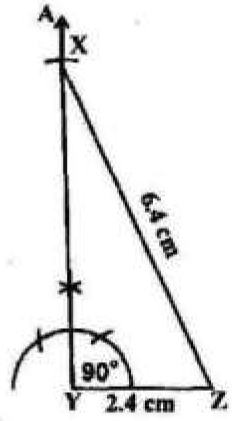
(i) Draw a line segment $m\overline{XY} = 6.1$ cm
(ii) At the end point X of $\overline{XY}$ make $m\angle YXZ = 90^\circ$.
(iii) With center at Y draw an arc of length 7.6cm which cuts $\overline{XA}$ at point Z.
(iv) Join Z with Y.
(v) Thus XYZ is the required triangle.

(ii) $m\overline{ZX} = 6.4$ cm, $m\overline{YZ} = 2.4$ cm and $m\angle Y = 90^\circ$

Solution: Given:

$m\overline{ZX} = 6.4cm$, $m\overline{YZ} = 2.4$ cm, $m\angle Y = 90^\circ$

Required: To construct the ΔXYZ by using given information.



Construction:

(i) Draw a line segment $m\ \overline{YZ} = 2.4cm$
(ii) At the end point Y of $\overline{YZ}$ make $m\angle ZYX = 90^\circ$.
(iii) With center at Z draw an arc of 6.4cm which meets $\overline{YA}$ at point X.
(iv) Join X with Z.

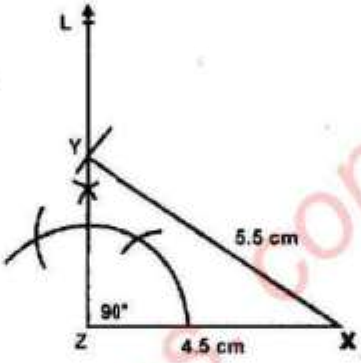
(v) Thus XYZ is the required triangle.

(iii) $m\overline{XY} = 5.5$ cm, $m\overline{ZX} = 4.5$ cm and $m\angle Z = 90^0$

Sol: Given:

$m\overline{XY} = 5.5$ cm, $m\overline{ZX} = 4.5$ cm and $m\angle Z = 90^0$

Required: To construct the ΔXYZ by using given information.



Construction:

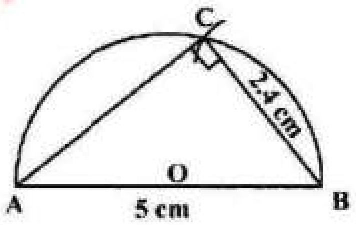
(i) Draw a line segment $m\overline{ZX} = 4.5$cm
(ii) At the end point Z of $\overline{ZX}$ make $m\angle XZL = 90^0$.
(iii) With center at X, draw an arc of radius 5.5cm which meets $\overline{ZL}$ at point Y.
(iv) Join X with Y.
(v) Thus, XYZ is the required triangle.

3. Construct a right-angled Δ measure of whose hypotenuse is 5 cm and one side is 3.2 cm. (Hint: Angle in a semi-circle is a right angle).

Solution: Given:

Hypotenuse = 5 cm, one side = 3.2 cm

Required: To construct right angled triangle using given data.



Construction:

(i) Draw line segments $m\overline{AB} = 5$cm
(ii) Draw a semicircle from O having radius $\overline{OA}$ or $\overline{OB}$.
(iii) With center at B, draw an arc of radius 3.2 cm which cut the semicircle at point C.
(iv) Join C with A and B.
(v) Thus ABC is the required right angled triangle.

4. Construct a right-angled isosceles triangle whose hypotenuse is

(i) 5.2 cm long (ii) 4.8 cm (iii) 6.2 cm (iv) 5.4 cm

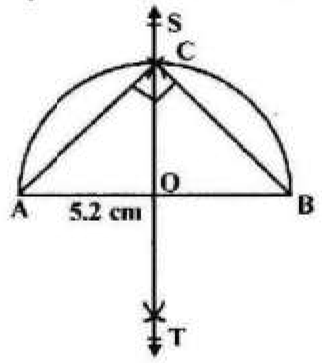
[Hint: A point on the right bisector of a line segment is equidistant from its end points.]

Solution (i): Given:

Hypotenuse = 5.2 cm,

Required:

To construct right angled triangle using given data.



Construction:

(i) Draw line segments $m\overline{AB} = 5.2$ cm
(ii) Draw a bisector of line $\overline{AB}$ at a point O on line $\overline{AB}$, such that

$$m\overline{OA} \cong m\overline{OB}.$$

(iii) Draw a semicircle from O having radius $\overline{OA}$ or $\overline{OB}$.

(iv) Semicircle cut the $\overleftrightarrow{ST}$ line at point C.

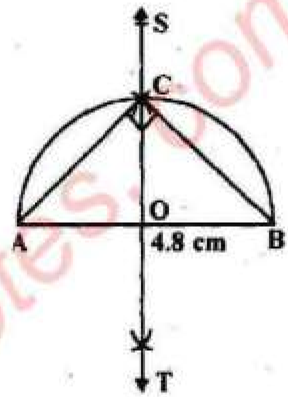
(v) Join C with A and B.

(vi) Thus ABC is the required right angled triangle.

Solution (ii):

Given: Hypotenuse = 4.8 cm,

Required: To construct right angled triangle using given data.



Construction:

(i) Draw line segments $m\overline{AB} = 4.8$ cm

(ii) Draw a bisector of line $\overline{AB}$ at a point O on line $\overline{AB}$, such that

$$m\overline{OA} \cong m\overline{OB}.$$

(iii) Draw a semicircle from O having radius $\overline{OA}$ or $\overline{OB}$.

(iv) Semicircle cut the $\overleftrightarrow{ST}$ line at point C.

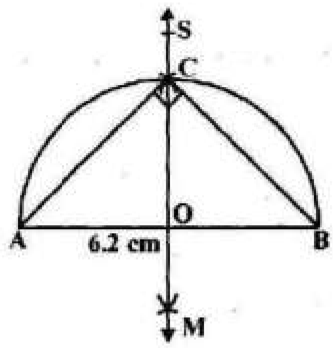
(v) Join C with A and B.

(vi) Thus ABC is the required right angled triangle.

Solution (iii):

Given: Hypotenuse = 6.2 cm,

Required: To construct right angled triangle using given data.



Construction:

(i) Draw line segments $m\overline{AB} = 6.2$ cm

(ii) Draw a bisector of line $\overline{AB}$ at a point O on line $\overline{AB}$, such that

$$m\overline{OA} \cong m\overline{OB}.$$

(iii) Draw a semicircle from O having radius $\overline{OA}$ or $\overline{OB}$.

(iv) Semicircle cut the $\overleftrightarrow{SM}$ line at point C.

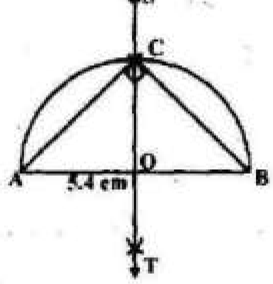
(v) Join C with A and B.

(vi) Thus ABC is the required right angled triangle.

Solution (iv):

Given: Hypotenuse = 5.4 cm,

Required: To construct right angled triangle using given data.



Construction:

(i) Draw line segments $m\overline{AB} = 5.4$ cm

(ii) Draw a bisector of line $\overline{AB}$ at a point O on line $\overline{AB}$, such that

$$m\overline{OA} \cong m\overline{OB}.$$

(iii) Draw a semicircle from O having radius $\overline{OA}$ or $\overline{OB}$.

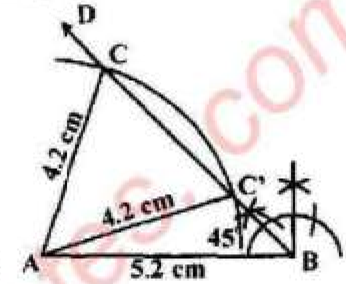
(iv) Semicircle cut the $\overleftrightarrow{ST}$ line at point C.

(v) Join C with A and B.

(vi) Thus ABC is the required right angled triangle.

5. (Ambiguous Case) Construct a ΔABC in which

(i) $m\overline{AC} = 4.2\text{ cm}$, $m\overline{AB} = 5.2\text{ cm}$, $m\angle B = 45^\circ$ (two Δs)



Solution: Given:

$m\overline{AC} = 4.2\text{ cm}$, $m\overline{AB} = 5.2\text{ cm}$, $m\angle B = 45^\circ$

Required:

To construct right angled triangle using given data.

Construction:

(i) Draw a straight line $m\overline{AB} = 5.2\text{ cm}$

(ii) At the end point B of $\overline{AB}$, make $m\angle ABD = 45^\circ$.

(iii) With center at A, draw an arc of radius 4.2 cm, which cuts $\overline{BD}$ at C and C′.

(iv) Join A with C and C′.

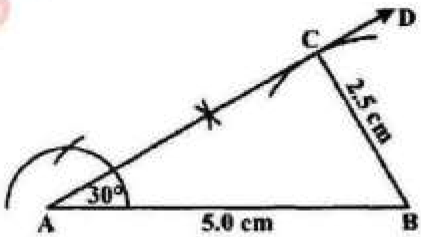
(v) Thus ΔABC and $\Delta ABC'$ are formed.

(ii) $m\overline{BC} = 2.5\text{ cm}$, $m\overline{AB} = 5.0\text{ cm}$, $m\angle A = 30^\circ$ (one Δ)

Solution: Given:

$m\overline{BC} = 2.5\text{ cm}$, $m\overline{AB} = 5.0\text{ cm}$, and $m\angle A = 30^\circ$ (one Δ).

Required: To construct the ΔABC using given information.



Construction:

(i) Draw a line segment $m\overline{AB} = 5.0\text{ cm}$

(ii) At the end point A of $\overline{AB}$, make $m\angle BAD = 30^\circ$.

(iii) With centre at B, draw an arc of radius 2.5cm which cuts the $\overline{AD}$ at point C.

(iv) Join B with C.

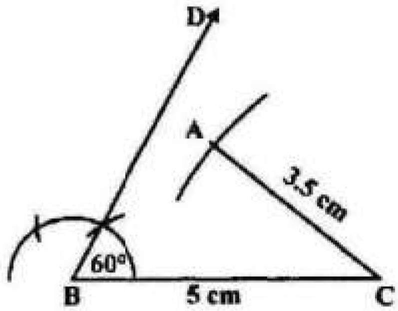
Thus ABC is the required triangle.

(iii) $m\overline{BC} = 5\text{ cm}$, $m\overline{AC} = 3.5\text{ cm}$, $m\angle B = 60^\circ$

Solution: Given:

$m\overline{BC} = 5\text{ cm}$, $m\overline{AC} = 3.5\text{ cm}$, $m\angle B = 60^\circ$.

Required: To construct right angled triangle using given data.



Construction:

(i) Draw a straight line $m\overline{BC} = 5\text{ cm}$

(ii) At the end point B of $\overline{AB}$, make $m\angle ABD = 60^\circ$.

(iii) With center at C, draw an arc of radius 3.5 cm, which does not cuts $\overrightarrow{BD}$ at any point.

(iv) Thus there will be no triangle in this case.

Point of concurrency:

Three or more than three lines are said to be concurrent, if they all pass through the same point. The common point is called the point of concurrency of the lines.

Importance of Point of concurrency:

The point of concurrency has its own importance in geometry. They are given special names.

(i) The internal bisectors of the angles of a triangle meet at a point called the **incentre** of the triangle.

(ii) The point of concurrency of the three perpendicular bisectors of the sides of a Δ is called the **circumcentre** of the Δ.

(iii) The point of concurrency of the three altitudes of a Δ is called its **orthocenter**.

(iv) The point where the three medians of a Δ meet is called the **centroid** of the triangle.

Drawing angle bisectors, altitudes:

(a) Draw angle bisectors of a given triangle and verify their concurrency.

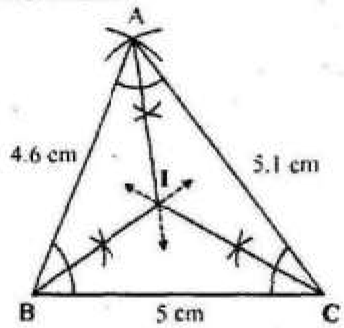
Example:

(i) Construct a ΔABC having given $m\overline{AB} = 4.6$ cm, $m\overline{BC} = 5$ cm and $m\overline{CA} = 5.1$ cm.

(ii) Draw its angle bisectors and verify that they are concurrent.

Solution: Given:

The side $m\overline{AB} = 4.6$ cm, $m\overline{BC} = 5$ cm and $m\overline{CA} = 5.1$ cm of a ΔABC.



Required:

(i) To construct ΔABC.

(ii) To draw its angle bisectors and verify their concurrency.

Construction:

(i) Take $m\overline{BC} = 5$ cm.

(ii) With B as centre and radius $m\overline{BA} = 4.6$ cm draw an arc.

(iii) With C as centre and radius $m\overline{CA} = 5.1$ cm draw another arc which intersects the first arc at A.

(iv) Join $\overline{BA}$ and $\overline{CA}$ to complete the ΔABC.

(v) Draw bisectors of $\angle B$ and $\angle C$ meeting each other in the point I.

(vi) Now draw bisector of the third $\angle A$.

(vii) We observe that the third angle bisector also passes through the point I.

(viii) Hence the angle bisectors of the ΔABC are concurrent at I.

(b) Draw altitudes of a given triangle and verify their concurrency:

Example:

(i) Construct a triangle ABC in which $m\overline{BC} = 5.9$ cm, $m\angle B = 56^\circ$ and $m\angle C = 44^\circ$.

(ii) Draw the altitudes of the triangle and verify that they are concurrent.

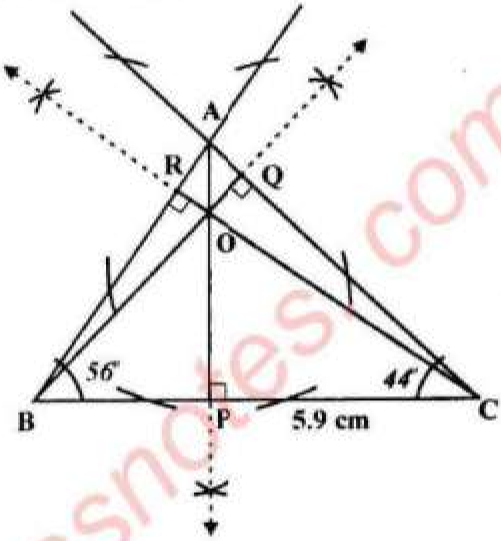
Solution: Given:

The side $m\overline{BC} = 5.9$ cm and $m\angle B = 56^0$ $m\angle C = 44^0$.

Required:

(i) To construct the ΔABC.

(ii) To draw its altitudes and verify their concurrency.



Construction:

(i) Take $m\overline{BC} = 5.9$ cm.

(ii) Draw $m\angle CBA = 56°$ and $m\angle BCA = 44°$ to complete the ΔABC.

(iii) From the vertex A drop $\overline{AP} \perp \overline{BC}$.

(iv) From the vertex B drop $\overline{BQ} \perp \overline{CA}$. These two altitudes meet in the point O inside the ΔABC.

(v) Now from the third vertex C, drop $\overline{CR} \perp \overline{AB}$.

(vi) We observe that this third altitude also passes through the point of intersection O of the first two altitudes.

(vii) Hence the three altitudes of ΔABC are concurrent at O.

(c) Draw perpendicular bisectors of the sides of a given triangle and verify their concurrency.

Example:

(i) Construct a ΔABC having given $m\overline{AB} = 4$ cm, $m\overline{BC} = 4.8$ cm and $m\overline{AC} = 3.6$ cm.

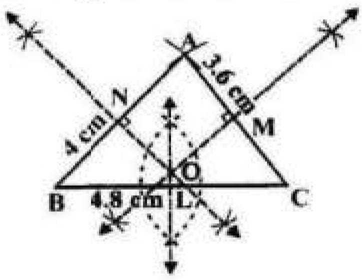
(ii) Draw perpendicular bisectors of its sides and verify that they are concurrent.

Solution: Given:

Three sides $m\overline{AB} = 4$ cm,
$m\overline{BC} = 4.8$ cm and $m\overline{AC} = 3.6$ cm of a ΔABC.

Required: **(i)** To construct the ΔABC.

(ii) To draw perpendicular bisectors of its sides and to verify that they are concurrent.



Construction:

(i) Take $m\overline{BC} = 4.8$ cm.

(ii) With B as centre and radius $m\overline{BA} = 4$ cm draw an arc.

(iii) With C as centre and radius $m\overline{CA} = 3.6$ cm draw another arc that intersects the first arc at A.

(iv) Join $\overline{BA}$ and $\overline{CA}$ to complete the ΔABC.

(v) Draw perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ meeting each other at the point O.

(vi) Now draw the perpendicular bisector of third side $\overline{AB}$.

(vii) We observe that it also passes through O, the point of intersection of first two perpendicular bisectors,

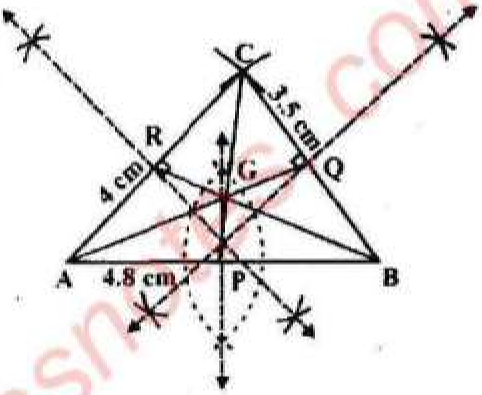
(viii) Hence the three perpendicular bisectors of ΔABC are concurrent at O.

(d) Draw medians of a given triangle and verify their concurrency

Example:

(i) Construct a ΔABC in which $m\overline{AB} = 4.8$ cm $m\overline{BC} = 3.5$ cm and $m\overline{AC} = 4$ cm.

(ii) Draw medians of ΔABC and verify that they are concurrent at a point within the triangle. By measurement show that the medians divide each other in the ratio 2:1.



Solution: Given:

Three sides $m\overline{AB}$=4.8 cm, $m\overline{BC}$=3.5 cm and $m\overline{AC}$ = 4 cm of a ΔABC.

Required: (i) Construct the ΔABC.

(ii) Draw its medians and verify their concurrency.

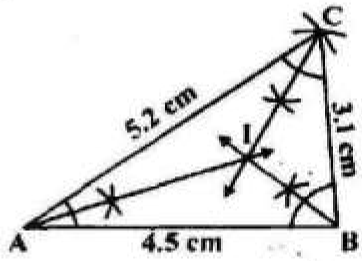
Construction:

(i) Take $m\overline{AB}$ = 4.8 cm.
(ii) With A as centre and $m\overline{AC}$ = 4 cm as radius draw an arc.
(iii) With B as centre and radius $m\overline{BC}$ = 3.5 cm draw another arc which intersects the first arc at C.
(iv) Join $\overline{AC}$ and $\overline{BC}$ to get the ΔABC.
(v) Draw perpendicular bisectors of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ of the ΔABC and mark their mid-points P, Q and R respectively.
(vi) Join A to the mid-point Q to get the median $\overline{AQ}$.
(vii) Join B to the mid-point R to have the median $\overline{BR}$.
(viii) The medians $\overline{AQ}$ and $\overline{BR}$ meet in the point G.
(ix) Now draw the third median $\overline{CP}$.
(x) We observe that the third median also passes through the point of intersection G of the first two medians.
(xi) Hence the three medians of the ΔABC pass through the same point G. That is, they are concurrent at G.

Solved Exercise 17.2

1. Construct the following Δ's ABC. Draw the bisectors of their angles and verify their concurrency.

(i) $m\overline{AB} = 4.5$ cm, $m\overline{BC} = 3.1$ cm and $m\overline{CA} = 5.2$ cm



Solution: Given:

The sides $m\overline{AB}$ = 4.5 cm, $m\overline{BC}$ = 3.1 cm and $m\overline{CA}$ = 5.2 cm of a ΔABC.

Required: (i) To construct ΔABC.

(ii) To draw its angle bisectors and verify heir concurrency.

Construction:

(i) Take $m\overline{AB}$ = 4.5 cm.

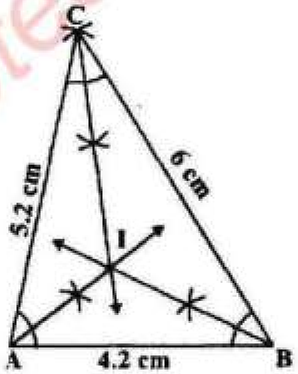
(ii) With B as centre and radius $m\overline{BC} = 3.1$ draw an arc.
(iii) With A as centre and radius $m\overline{CA} = 5.2$ cm draw another arc which intersects the first arc at C.
(iv) Join $\overline{BC}$ and $\overline{AC}$ to complete the ΔABC.
(v) Draw bisectors of $\angle A$ and $\angle B$ meeting each other in the point I.
(vi) Now draw bisector of the third $\angle C$.
(vii) We observe that the third angle bisector also passes through the point I.
(viii) Hence the angle bisectors of the ΔABC are concurrent at I.

(ii) $m\overline{AB} = 4.2$ **cm,** $m\overline{BC} = 6$ **cm and** $m\overline{CA} = 5.2$ **cm**

Solution: Given:

The sides $m\overline{AB} = 4.2$ cm, $m\overline{BC} = 6$ cm and $m\overline{CA} = 5.2$ cm of a ΔABC.

Required:
(i) To construct ΔABC.
(ii) To draw its angle bisectors and verify their concurrency.



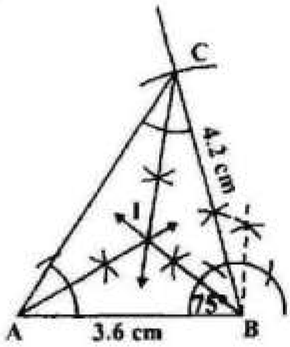
Construction:
(i) Take $m\overline{AB} = 4.2$ cm.
(ii) With B as centre and radius $m\overline{BC} = 6$cm draw an arc.
(iii) With A as centre and radius $m\overline{AC} = 5.2$ cm draw another arc which intersects the first arc at C.
(iv) Join BC and AC to complete the ΔABC.
(v) Draw bisectors of $\angle A$ and $\angle B$ meeting each other in the point I.
(vi) Now draw bisector of the third $\angle C$.
(vii) We observe that the third angle bisector also passes through the point I.
(viii) Hence the angle bisectors of the ΔABC are concurrent at I.

(iii) $m\overline{AB} = 3.6$ **cm,** $m\overline{BC} = 4.2$ **cm and** $m\angle B = 75°$

Solution: Given:

The sides $m\overline{AB} = 3.6$ cm, $m\overline{BC} = 4.2$ cm and $m\angle B = 75°$ of a ΔABC.

Required: (i) To construct ΔABC.
(ii) To draw its angle bisectors and verify their concurrency.



Construction:
(i) Take $m\overline{AB} = 3.6$ cm.
(ii) With B as centre draw $m\angle B = 75°$.
(iii) With B as centre and radius $m\overline{BC} = 4.2$ cm draw an arc.
(iv) Join A with C to complete the ΔABC.
(v) Draw bisectors of $\angle A$ and $\angle B$ meeting each other in the point I.
(vi) Now draw bisector of the third $\angle C$.
(vii) We observe that the third angle bisector also passes through the point I.
(viii) Hence the angle bisectors of the ΔABC are concurrent at I.

2. Construct the following Δ's PQR. Draw their altitudes and show that they are concurrent.

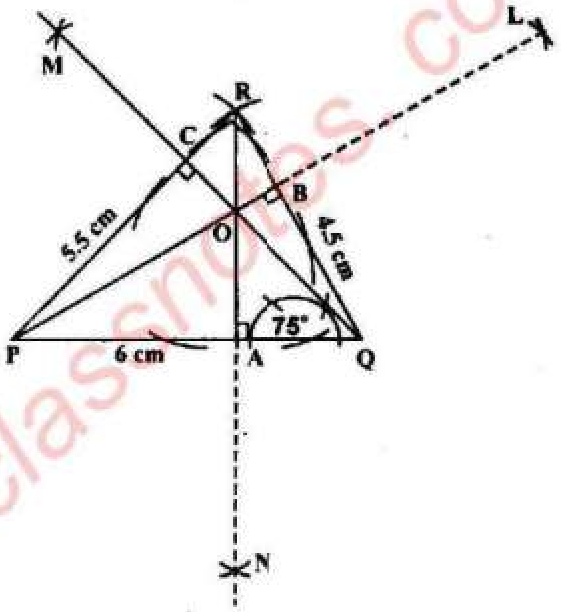
(i) $m\overline{PQ} = 6cm$, $m\overline{QR} = 4.5cm$ and $m\overline{PR} = 5.5\ cm$

Solution: Given: The sides $m\overline{PQ} = 6$ cm, $m\overline{QR} = 4.5$cm and $m\overline{PR} = 5.5$ cm

Required: **(i)** To construct the ΔPQR.

(ii) To draw its altitudes and verify their concurrency.

Construction:



(i) Take $m\overline{PQ} = 6$ cm.

(ii) With P as centre and radius $m\overline{PR} = 5.5$ draw an arc. With Q as centre and radius $m\overline{QR} = 4.5$ cm draw another arc which intersects the first arc at R. Join $\overline{PR}$ and $\overline{QR}$ to complete the ΔPQR.

(iii) From the vertex P drop $\overline{PL} \perp \overline{QR}$.

(iv) From the vertex Q drop $\overline{QM} \perp \overline{PR}$. These two altitudes meet in the point O inside the ΔPQR.

(v) Now from the third vertex E, drop $\overline{RN} \perp \overline{PQ}$.

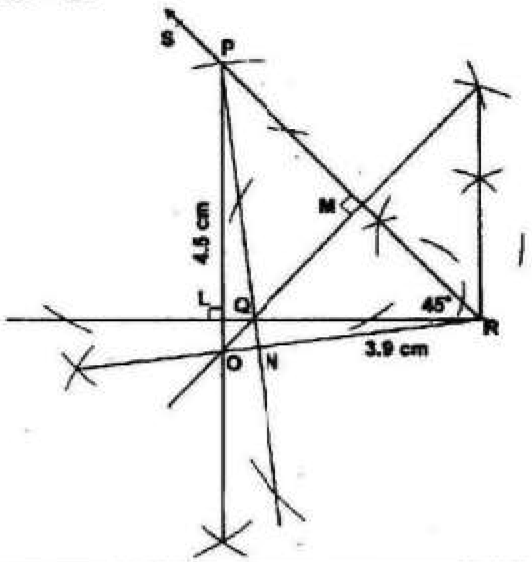
(vi) We observe that the third altitude also passes through the point of intersection O of the first two altitudes.

(vii) Hence the three altitudes of ΔPQR are concurrent at O.

(ii) $m\overline{PQ} = 4.5\ cm$, $m\overline{QR} = 3.9\ cm$ and $m\angle R = 45°$

Solution: Given:

The sides $m\overline{PQ} = 4.5$ cm, $m\overline{QR} = 3.9$cm and $m\angle R = 45°$



Required:

(i) To construct the ΔPQR.

(ii) To draw its altitudes and verify their concurrency.

Construction:

(i) Take $m\overline{QR} = 3.9$ cm.

(ii) At point R make $m\angle QRS = 45°$. With Q as centre draw an arc of radius 4.5 cm which intersects $\overline{RS}$ at the point P. Join P to Q to complete the ΔPQR.

(iii) From the vertex P drop $\overline{PL} \perp \overline{QR}$.

(iv) From the vertex Q drop $\overline{QM} \perp \overline{PR}$. These two altitudes meet in the point O inside the ΔPQR.

(v) Now from the third vertex E, drop $\overline{RN} \perp \overline{PQ}$.

(vi) We observe that this third altitude also passes through the point of intersection O of the first two altitudes.

(vii) Hence the three altitudes of ΔPQR are concurrent at O.

(iii) $m\overline{RP} = 3.6$ **cm,** $m\angle Q = 30°$ **and** $m\angle P = 105°$

Solution:

$m\angle Q = 30°, m\angle P = 105°$

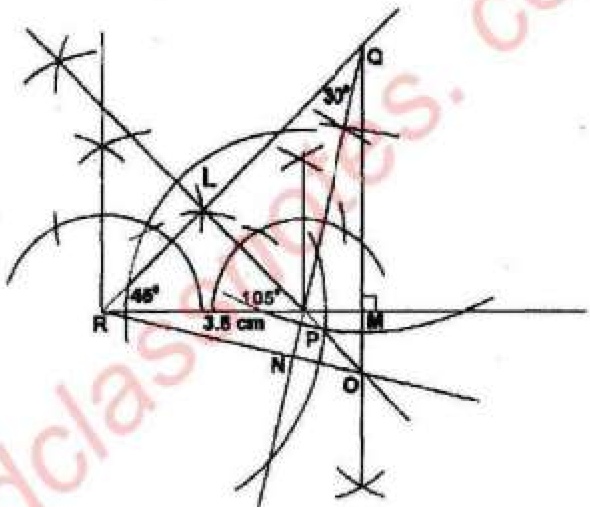
But $m\angle P + m\angle Q + m\angle R = 180°$

$m\angle R = 180° - (m\angle P + m\angle Q)$

$m\angle R = 180° - (105° + 30°)$

$m\angle R = 180° - 135°$

$m\angle R = 45°$



Given: The side mRP = 3.6 cm, $m\angle Q = 30°$ and $m\angle P = 105°$

Required:

(i) To construct the ΔPQR.

(ii) To draw its altitudes and verify their concurrency.

Construction:

(i) Take $m\overline{RP} = 3.6$ cm.

(ii) Draw $m\angle PRQ = 45°$ and $m\angle RPQ = 105°$. Join Q to R and P to complete the ΔPQR.

(iii) From the vertex P drop $\overline{PL} \perp \overline{QR}$.

(iv) From the vertex Q drop $\overline{QM} \perp \overline{PR}$. These two altitudes meet in the point O outside the ΔPQR.

(v) Now from the third vertex R, drop $\overline{RN} \perp \overline{PQ}$.

(vi) We observe that this third altitude also passes through the point of intersection O of the first two altitudes.

(vii) Hence the three altitudes of ΔPQR are concurrent at O.

3. Construct the following triangles ABC. Draw the perpendicular bisectors of their sides and verify their concurrency. Do they meet inside the triangle?

(i) $m\overline{AB} = 5.3$ **cm,** $m\angle A = 45°$**,** $m\angle B = 30°$

Solution: Given: $m\overline{AB} = 5.3$ cm,

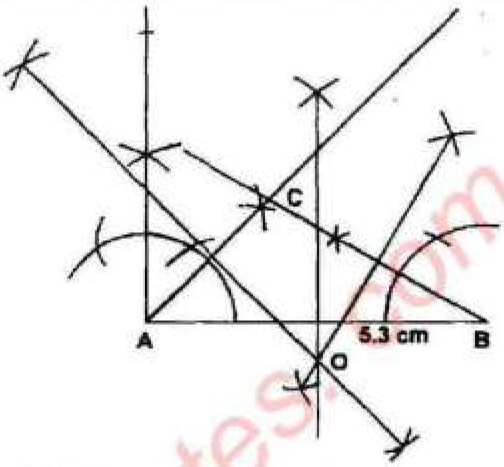
$m\angle A = 45°, m\angle B = 30°$

Required: (i) To construct the ΔABC.

(ii) To draw perpendicular bisectors of its sides and to verify that they are concurrent.

Construction:

(i) Take $m\overline{AB} = 5.3$ cm.



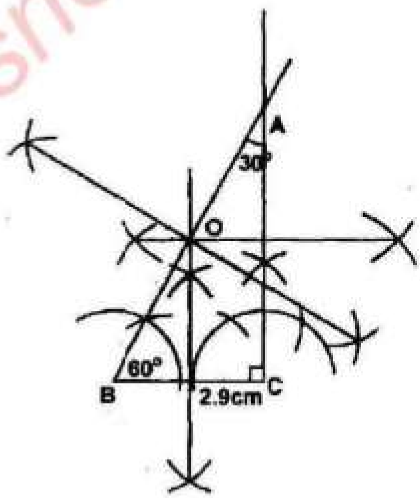
(ii) Draw $m\angle BAC = 45^\circ$ at point A.
(iii) Draw $m\angle ABC = 30^\circ$ at point B.
(iv) The terminal sides of these angles meet at C to complete the ΔABC.
(v) Draw perpendicular bisectors of BC and CA meeting each other at the point O.
(vi) Now draw the perpendicular bisector of third side $\overline{AB}$.
(vii) We observe that it also passes through O, the point of intersection of first two perpendicular bisectors,
(viii) Hence the three perpendicular bisectors of ΔABC are concurrent at O.

(ii) $\mathbf{m\overline{BC} = 2.9\ cm, m\angle A = 30^\circ\ , m\angle B = 60^\circ}$

Solution: $m\angle A = 30^\circ, m\angle B = 60^\circ$

But $m\angle A + m\angle B + m\angle C = 180^\circ$

$$m\angle C = 180^\circ - (m\angle A + m\angle B)$$
$$m\angle C = 180^\circ - (30^\circ + 60^\circ)$$
$$m\angle C = 180^\circ - 90^\circ$$
$$m\angle C = 90^\circ$$



Given: $m\overline{BC} = 2.9$ cm,
$m\angle A = 30^\circ, m\angle B = 60^\circ$ and
$m\angle C = 90^\circ$

Required: **(i)** To construct the ΔABC.
(ii) To draw perpendicular bisectors of its sides and to verify that they are concurrent.

Construction:
(i) Take $m\overline{BC} = 2.9$ cm.
(ii) Draw $m\angle ABC = 60^\circ$ at point B.
(iii) Draw $m\angle ACB = 90^\circ$ at point C.
(iv) The terminal sides of these angles meet at A to complete the ΔABC.
(v) Draw perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ meeting each other at the point O.
(vi) Now draw the perpendicular bisector of third side $\overline{AB}$.
(vii) We observe that it also passes through O, the point of intersection of first two perpendicular bisectors,
(viii) Hence the three perpendicular bisectors of ΔABC are concurrent at O.

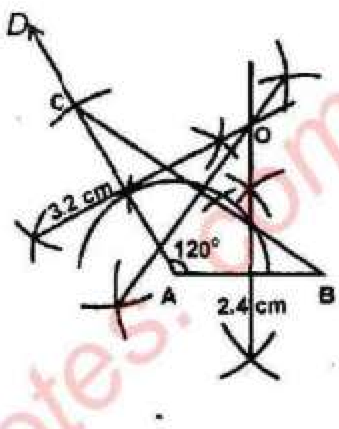
(iii) $\mathbf{m\overline{AB} = 2.4\ cm,\quad m\overline{AC} = 3.2\ cm,\quad m\angle A = 120^\circ}$

Solution: Given: $m\overline{AB} = 2.4$ cm,
$mAC = 3.2, m\angle A = 120^\circ$

Required: **(i)** To construct the ΔABC.
(ii) To draw perpendicular bisectors of its sides and to verify that they are

concurrent.

Construction:



(i) Take $m\overline{AB} = 2.4$ cm.

(ii) Draw $m\angle BAC = 120^\circ$ at point A.

(iii) With centre at the point A and radius $m\overline{AC}$ =3.2cm, draw an arc which intersects $\overline{AD}$ at point C.

(iv) Join B to C to complete the ΔABC.

(v) Draw perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ meeting each other at the point O.

(vi) Now draw the perpendicular bisector of third side $\overline{AB}$.

(vii) We observe that it also passes through O, the point of intersection of first two perpendicular bisectors,

(viii) Hence the three perpendicular bisectors of ΔABC are concurrent at O.

4. Construct the following Δs XYZ. Draw their three medians and show that they are concurrent.

(i) $m\overline{YZ} = 4.1$cm, $m\angle Y = 60^\circ$ and $m\angle X = 75^\circ$

Solution: $m\angle X = 75^\circ$, $m\angle Y = 60^\circ$

But $m\angle X + m\angle Y + m\angle Z = 180^\circ$

$m\angle Z = 180^\circ - (m\angle X + m\angle Y)$

$m\angle Z = 180^\circ - (75^\circ + 60^\circ)$

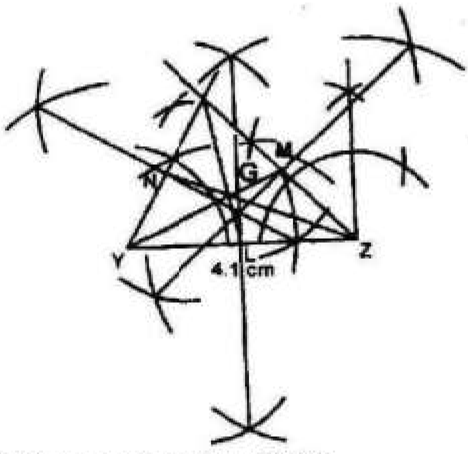
$m\angle Z = 180^\circ - 135^\circ$

$m\angle Z = 45^\circ$

Given: $m\overline{YZ}$=4.1cm, $m\angle Y$=60° and $m\angle X$=75°

Required: (i) Construct the ΔXYZ.

(ii) Draw its medians and verify their concurrency.



Construction:

(i) Take $m\overline{YZ} = 4.1$ cm.

(ii) Draw $m\angle XYZ = 60^\circ$ at point Y.

(iii) Draw $m\angle XZY = 45^\circ$ at point Z.

(iv) The terminal sides of the two angles meet at X to complete the ΔXYZ.

(v) Join X to the mid-point L to get the median $\overline{XL}$.

(vi) Join Y to the mid-point M to have the median $\overline{YM}$.

(vii) The medians $\overline{XL}$ and $\overline{YM}$ meet in the point G.

(viii) Now draw the third median $\overline{ZN}$.

(ix) We observe that the third median also passes through the point of intersection G of the first two medians.

(x) Hence the three medians of the ΔXYZ pass through the same point G. That is, they are concurrent at G.

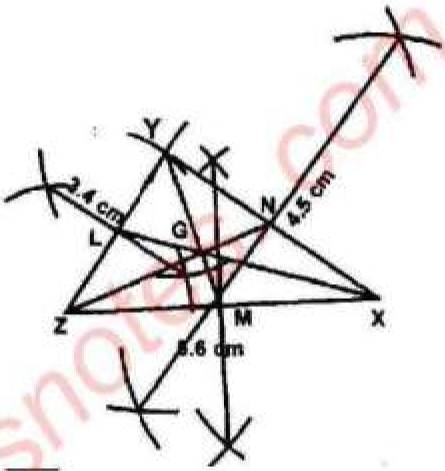
(ii) $m\overline{XY} = 4.5$ **cm,** $m\overline{YZ} = 3.4$ **cm and** $m\overline{ZX} = 5.6$ **cm**

Solution: Given:

Three sides mXY = 4.5 cm, mYZ = 3.4 cm and mZX = 5.6 of a ΔXYZ.

Required: (i) Construct the ΔXYZ.

(ii) Draw its medians and verify their concurrency.



Construction:

(i) Take $m\overline{ZX} = 5.6$ cm.

(ii) With Z as centre and radius $m\overline{ZY} = 3.4$cm draw an arc.

(iii) With X as centre and radius $m\overline{XY} = 4.5$cm draw another arc which intersects the first arc at Y.

(iv) Join $\overline{ZY}$ and $\overline{XY}$ to complete the ΔXYZ.

(v) Join X to the mid-point L to get the median $\overline{XL}$.

(vi) Join Y to the mid-point M to have the median $\overline{YM}$.

(vii) The medians $\overline{XL}$ and $\overline{YM}$ meet in the point G.

(viii) Now draw the third median $\overline{ZN}$.

(ix) We observe that the third median also passes through the point of intersection G of the first two medians.

(x) Hence the three medians of the ΔXYZ pass through the same point G. That is, they are concurrent at G.

(iii) $m\overline{ZX} = 4.3$**cm,** $m\angle X = 75^0$**, and** $m\angle Y = 45°$

Solution: $m\angle X = 75°$, $m\angle Y = 45°$

But $m\angle X + m\angle Y + m\angle Z = 180°$

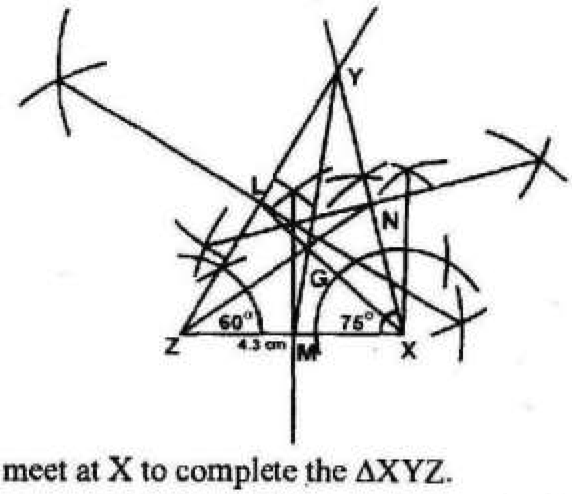
$$m\angle Z = 180° - (m\angle X + m\angle Y)$$
$$m\angle Z = 180° - (75° + 45°)$$
$$m\angle Z = 180° - 120°$$
$$m\angle Z = 60°$$

Given: $m\overline{ZX} = 4.3$ cm, $m\angle X = 75°$ and $m\angle Y = 60°$

Required:

(i) Construct the ΔXYZ.

(ii) Draw its medians and verify their concurrency.



Construction:

(i) Take $m\overline{ZX} = 4.3$ cm.

(ii) Draw $m\angle XZY = 60°$ at point Z.

(iii) Draw $m\angle YXZ = 75°$ at point X.

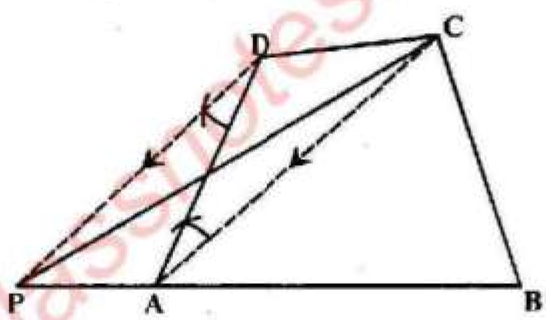
(iv) The terminal sides of the two angles meet at X to complete the ΔXYZ.

(v) Join X to the mid-point L to get the median $\overline{XL}$.
(vi) Join Y to the mid-point M to have the median $\overline{YM}$.
(vii) The medians $\overline{XL}$ and $\overline{YM}$ meet in the point G.
(viii) Now draw the third median $\overline{ZN}$.
(ix) We observe that the third median also passes through the point of intersection G of the first two medians.
(x) Hence the three medians of the ΔXYZ pass through the same point G. That is, they are concurrent at G.

Figures with Equal Areas:

(I) Construct a triangle equal in area to a given quadrilateral:

Given: A quadrilateral ABCD
Required: To construct a Δ equal in area to quadrilateral ABCD.
Construction:
(i) Join $\overline{AC}$.
(ii) Through D draw $\overline{DP} \parallel \overline{CA}$, meeting $\overline{BA}$ produced at P.
(in) Join $\overline{PC}$.
(iv) Then PBC is the required triangle.



Observe that:

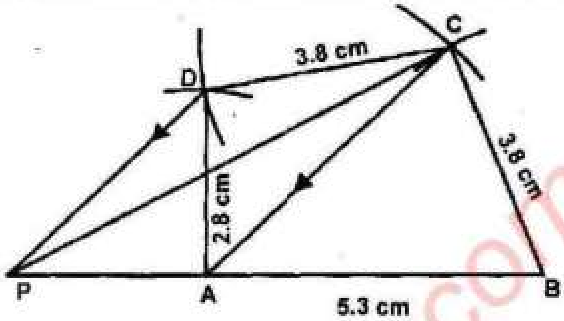
Δs APC, ADC stand on the same, base $\overline{AC}$ and between the same parallels $\overline{AC}$ and $\overline{PD}$.

Hence $\Delta APC = \Delta ADC$
$\Delta APC + \Delta ABC = \Delta ADC + \Delta ABC$
Or $\Delta PBC =$ quadrilateral ABCD.

Solved Exercise 17.3

1. (i) Construct a quadrilateral ABCD, having $m\overline{AB} = m\overline{AC} = 5.3$ cm, $m\overline{BC} = m\overline{CD} = 3.8$ cm and $m\overline{AD} = 2.8$ cm.
(ii) On the side BC construct a Δ equal in area to the quardilateral ABCD.

Solution: Given:
$m\overline{AB} = m\overline{AC} = 5.3$ cm, $m\overline{BC} = \overline{CD} = 3.8$ cm and $m\overline{AD} = 2.8$ cm
Required: **(i)** To consruct A quadrilateral ABCD.
(ii) On the side $\overline{BC}$ to construct a Δ equal in area to the quadrilateral ABCD.
Construction:
(i) Draw a line segment $m\overline{AB} = 5.3$ cm.
(ii) With centre at A and radius 5.3 cm draw an arc.
(iii) With centre at B and radius 3.8 cm draw another arc to cut the first arc at C.
(iv) Join $\overline{BC}$ and $\overline{AC}$.



(v) With centre at C and radius 3.8 cm draw an arc

(vi) With centre at A and radius 2.8 cm draw another arc to cut the first arc at D.

(vii) Join $\overline{AD}$ and $\overline{DC}$ to complete the quadrillateral ABCD.

(viii) Through D draw $\overline{DP} \parallel \overline{CA}$ meeting $\overline{BA}$ produced at P.

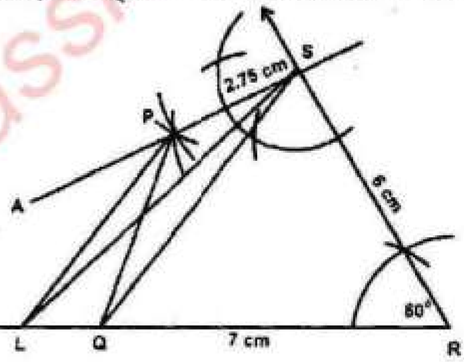
(ix) Join $\overline{PC}$.

(x) The ΔPBC is the required triangle.

2. Construct a Δ equal in area to the quadrilateral PQRS, having $m\overline{QR} = 7$ cm, $m\overline{RS} = 6$cm, $m\overline{SP} = 2.75$ cm, $m\angle QRS = 60^\circ$, and $m\angle RSP = 90^\circ$. [Hint: $2.75 = \frac{1}{2} \times 5.5$]

Solution: Given:

$m\overline{QR} = 7$ cm, $m\overline{RS} = 6$ cm, $m\overline{SP} = 2.75$ cm, $m\angle QRS = 60^\circ$ and mRSP $= 90^\circ$



Required: (i) To construct A quadrilateral PQRS.

(ii) On the side $\overline{RS}$ to construct a Δ equal in area to the quadrilateral PQRS.

Construction:

(i) Draw a line segment $m\overline{QR} = 7$cm.

(ii) With centre at R and radius $m\angle QRS = 60^\circ$

(iii) With centre at R cut off $\overline{RS} = 6$ cm.

(iv) At the end point S make $\angle RSA = 90^\circ$

(v) From $\overline{SA}$ cut off $m\overline{SP} = 2.75$ cm

(vi) Join $\overline{PQ}$ to complete the quadrilateral PQRS.

(vii) Join $\overline{QS}$.

(viii) Through P draw $\overline{PL} \parallel \overline{QS}$ meet $\overline{RQ}$ produced at L.

(ix) Join $\overline{SL}$.

(x) Then LRS is the required triangle.

3. Construct a Δ equal in area to the quadrilateral ABCD, having $m\overline{AB} = 6$ cm, $m\overline{BC} = 4$ cm, $m\overline{AC} = 7.2$ cm, $m\angle BAD = 105^\circ$, and $m\overline{BD} = 8$ cm.

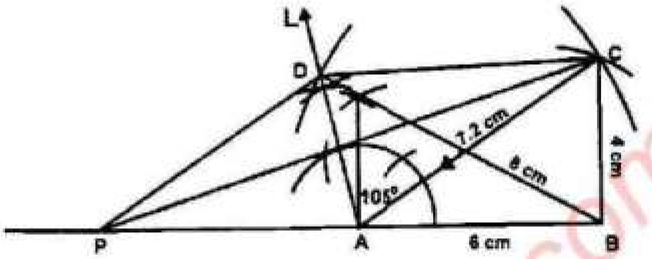
Solution: Given:

$m\overline{AB} = 6$cm, $m\overline{BC} = 4$cm, $m\overline{AC} = 7.2$ cm, $m\angle BAD = 105^\circ$ and $m\overline{BD} = 8$ cm of quadrilateral ABCD.

Required: (i) To construct A quadrilateral with given data.

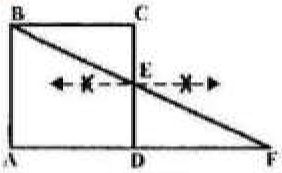
(ii) To construct a Δ equal in area to the quadrilateral ABCD

Construction:



(i) Take $m\overline{AB} = 6$ cm,
(ii) With centre at the end point A and radius 7.2 cm draw an arc.
(iii) With B as centre and radius 4 cm draw another arc to cut the first arc at C.
(iv) Join $\overline{AC}$ and $\overline{BC}$.
(v) At the end point A make $m\angle BAL = 105°$.
(vi) With B as centre and radius 8 cm draw an arc to cut $\overrightarrow{AL}$ at the point D.
(vii) Join $\overline{DC}$ to complete the quadrilateral ABCD.
(viii) Draw $\overline{DP} \parallel \overline{CA}$ to meet $\overline{BA}$ produced at P.
(ix) Join P to C.
(x) Then PBC is the required triangle.

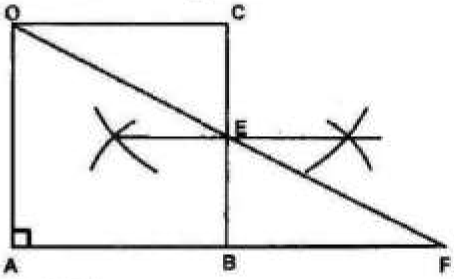
4. Construct a right-angled triangle equal in area to a given square.



Solution: Given
A square ABCD.

Required: To construct a right angled triangle equal in area to square ABCD.

Construction:



(i) Draw perpendicular bisector of $\overline{BC}$.
(ii) Mark point E the mid point of $\overline{BC}$.
(iii) Draw the straight line DEF to meet $\overline{AB}$ produced at the point F.
(iv) Then DAF is the required right angled triangle.

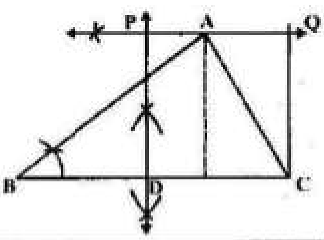
Observe that

Right angled triangles DCE and FBE are congruent because mCE = mBE and $m\angle CED = m\angle BEF$.

So area ΔDAF

= area quadrilateral ABED + area ΔFBE.
= area quadrilateral ABED + area ΔECD
= area square ABCD

(ii) Construct a rectangle equal in area to a given triangle:



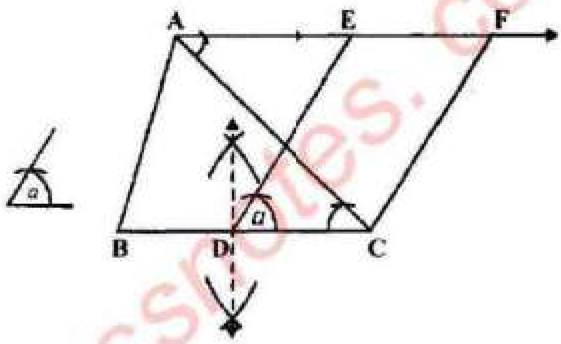
Given: ΔABC

Required: To construct a rectangle equal in area to ΔABC

Construction:

(i) Take a ΔABC.
(ii) Draw $\overleftrightarrow{DP}$, the perpendicular bisector of $\overline{BC}$.
(iii) Through, the vertex A of ΔABC draw $\overleftrightarrow{PAQ} \parallel \overline{BC}$ intersecting $\overleftrightarrow{PD}$ at P.
(iv) Take $m\overline{PQ} = m\overline{DC}$.
(v) Join Q and C.
(vi) Then CDPQ is the required rectangle.

Example: Construct a parallelogram equal in area to a given triangle having one angle equal to a, given angle.

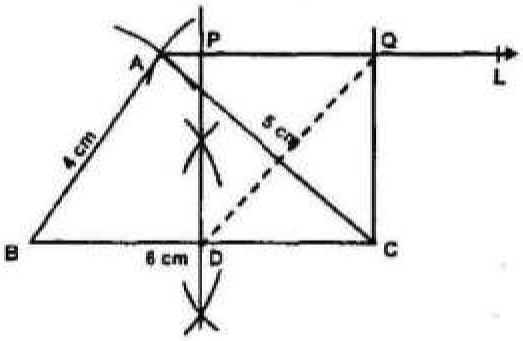


Given: ΔABC and $\angle \alpha$
Required: To construct a parallelogram equal in area to ΔABC and having one angle = $\angle\alpha$
Construction:
1. Bisect $\overline{BC}$ at D.
2. Draw $\overline{DE}$ making $\angle CDE = \angle\alpha$
3. Draw $\overline{AEF} \parallel \overline{BC}$ cutting $\overline{DE}$ at E.
4. Cut off $\overline{EF} = \overline{DC}$. Join C and F.
Then CDEF is the required parallelogram.

Solved Exercise 17.4

1. **Construct a Δ with sides 4 cm, 5 cm and 6 cm and construct a rectangle having its area equal to that of the Δ. Measure its diagonals. Are they equal?**

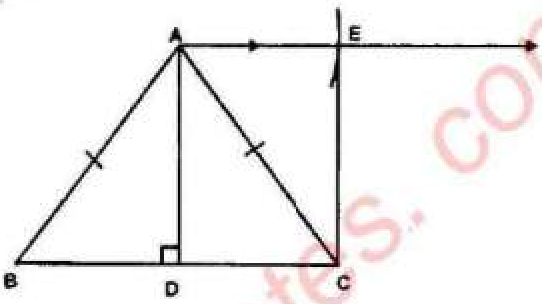
Given: Measures of 3 sides of a triangle as $m\overline{BC}$ =6cm, $m\overline{CA}$ =5cm, in $m\overline{AB}$ = 4cm.



Required
(i) To construct ΔABC.
(ii) To construct a rectangle equal in area to ΔABC.
(iii) To measure a diagonals of rectangle.
Construction
(i) Draw a, line segment $m\overline{BC} = 6$ cm.
(ii) With centre at the point B and radius as 4 cm draw an arc.
(iii) With centre at the point C with radius 5 cm draw another arc to cut the first arc at the point A.
(iv) Join $\overline{AB}$ and $\overline{AC}$ to complete the ΔABC.
(v) Bisect $\overline{BC}$ at D.
(vi) Draw $\overrightarrow{AL} \parallel \overline{BC}$.
(vii) Draw perpendicular $\overline{DP}$ to meet $\overrightarrow{AL}$ at P.
(viii) Cut off $\overline{PQ} = \overline{DC}$.

(ix) Join Q to C.
Then PQCD is the required rectangle. Measure the diagonals $\overline{DQ} = 4.5$ cm and $\overline{CP} = 4.5$cm.
Hence the diagonals $\overline{DQ}$ and $\overline{CP}$ are equal.

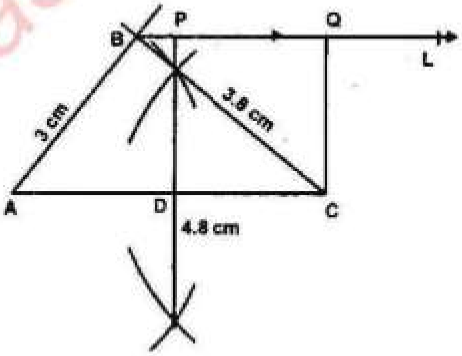
2. Transform an isosceles Δ into a rectangle.



Given: An isosceles ΔABC such that $m\overline{AB} = m\overline{AC}$.
Required: To construct a rectangle equal in area to ΔABC.
Construction
(i) Draw $\overline{AD} \perp \overline{BC}$.
(ii) Draw $\overline{AE} \parallel \overline{DC}$.
(iii) Cut off $m\overline{AE} = m\overline{DC}$.
(iv) Join $\overline{EC}$.
Then ADCE is the required rectangle.

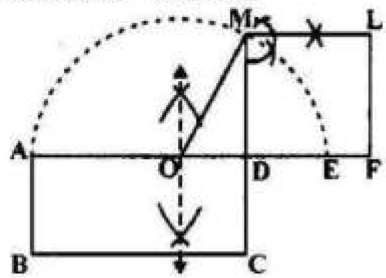
3. Construct a ΔABC such that $m\overline{AB} = 3$ cm, $m\overline{BC} = 3.8$ cm, $m\overline{AC} = 4.8$ cm. Construct a rectangle equal in area to the ΔABC, and measure its sides.

Given: Sides $m\overline{AB} = 3$ cm, $m\overline{BC} = 3.8$ cm, $m\overline{AC} = 4.8$ cm of ΔABC.
Required: (i) To construct ΔABC
(ii) To construct a rectangle equal in area to the ΔABC.
(iii) To measure its sides.



Construction
(i) Draw a line segment. $m\overline{AC} = 4.8$ cm.
(ii) With centre at A and radius 3 cm .draw an arc.
(iii) With centre at C and radius 3.8 cm draw another arc to cut the first arc at B.
(iv) Join $\overline{AB}$ and $\overline{BC}$ to complete the ΔABC.
(v) Draw $\overline{BL} \parallel \overline{AC}$.
(vi) Draw $\overline{DP}$ the perpendicular bisector of $\overline{AC}$ to meet $\overline{BL}$ at P.
(vii) Cut off $m\overline{PQ} = m\overline{DC}$.
(viii) Join $\overline{QC}$.
(ix) Then PQCD is the required rectangle.
(x) Measure the sides of the rectangle $m\overline{DC} = 2.4$ cm and $m\overline{DP} = 2.3$ cm.

(iii) Construct a square equal m area to a given rectangle.

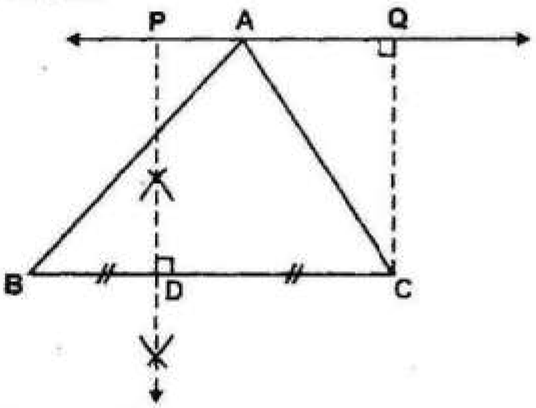


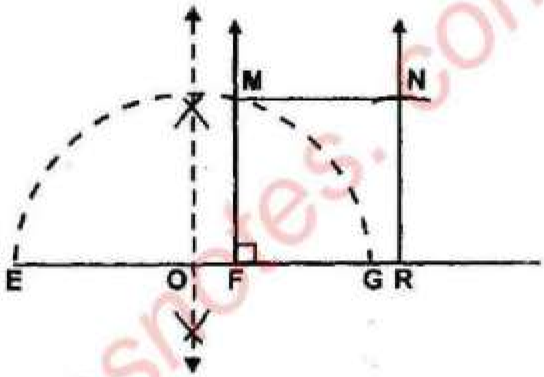
Given: A rectangle ABCD.
Required: To construct a square equal in area to rectangle ABCD.
Construction
(i) Produce $\overline{AD}$ to E making $m\overline{DE} = m\overline{CD}$.
(ii) Bisect $\overline{AE}$ at O.

(iii) With centre O and radius $\overline{OA}$ describe a semi-circle.
(iv) Produce $\overline{CD}$ to meet the semi-circle in M.
(v) On $\overline{DM}$ as a side, construct a square DFLM.
This shall be the required square.

Example: Construct a square equal in area to a given triangle.

Solution





Given: ΔABC.

Required: To construct a square equal in area to ΔABC.

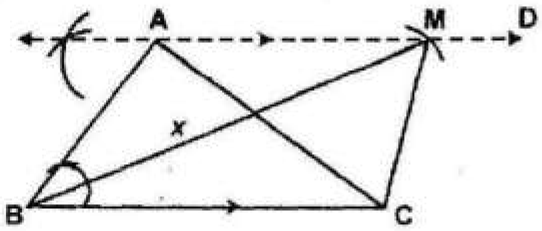
Construction

(i) Draw $\overleftrightarrow{PAQ} \parallel \overline{BC}$.
(ii) Draw perpendicular bisector of $\overline{BC}$, bisecting it at D and meeting $\overleftrightarrow{PAQ}$ at P.
(iii) Draw $\overline{CQ} \perp \overline{PQ}$ meeting it in Q.
(iv) Take a line EFG and cut off $\overline{EF} = \overline{DP}$ and $\overline{FG} = \overline{DC}$.
(v) Bisect $\overline{EG}$ at O.
(vi) With O as centre and radius = $\overline{OE}$ draw a semi-circle.
(vii) At F draw $\overline{FM} \perp \overline{EG}$ meeting the serai-circle at M.
(viii) With $\overline{MF}$ as a side, complete the required square FMNR.

(iv) Construct a triangle of equivalent area on a base of given length.

Given: ΔABC

Required: To construct a triangle with base x and having area equivalent to area ΔABC.



Construction

(i) Construct the given ΔABC.
(ii) Draw $\overline{AD} \parallel \overline{BC}$.
(iii) With B as centre and radius = x, draw an are cutting $\overline{AD}$ in M.
(iv) Join $\overline{BM}$ and $\overline{CM}$.
(v) Then BCM is the required triangle with base $\overline{BM}$ = x and area equivalent to area ΔBC.

○○○○○○○○○○○○○

Solved Exercise 17.5

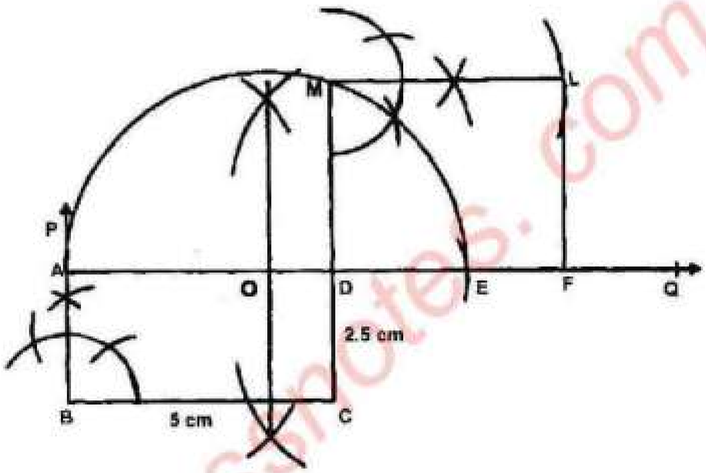
1. Construct a rectangle whose adjacent sides are 2.5 cm and 5 cm respectively. Construct a square having area equal to the given rectangle,

Given: $m\overline{BC} = 5$ cm and $m\overline{CD} = 2.5$ cm of a rectangle ABCD.

Required

(i) To construct rectangle ABCD.

(ii) To construct a square having area equal to that of rectangle ABCD.



Construction

(i) Draw the line segment $m\overline{BC} = 5$ cm.

(ii) Draw $\overline{BP} \perp \overline{BC}$.

(iii) Cut off $m\overline{BA} = 2.5$cm

(iv) Complete the rectangle ABCD with $m\overline{DC} = m\overline{AB}$ and $m\overline{AD} = m\overline{BC}$.

(v) Produce $\overline{AD}$ to Q

(vi) Cut off $m\overline{DE} = m\overline{DC}$

(vii) Bisect AE at the point O.

(viii) With O as the centre and $m\overline{OA}$ as radius draw semicircle.

(ix) Produce $\overline{CD}$ to meet the semi circle at M.

(x) With $m\overline{DM}$ as a side complete the square DFLM.

(xi) Then DFLM is the required square.

2. Construct a square equal in area to a rectangle whose adjacent sides are 4.5 cm and 2.2 cm respectively. Measure the sides of the square and find its area and compare with the area of the rectangle.

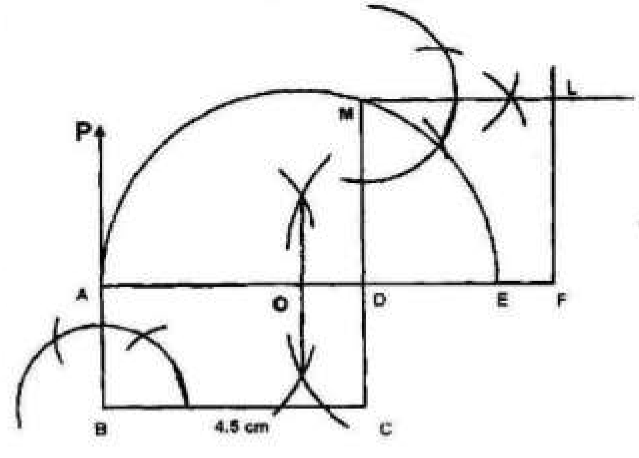
Given: Sides $m\overline{BC} = 4.5$ cm and $m\overline{DC} = 2.2$ cm of rectangle ABCD.

Required:

(i) To construct rectangle ABCD.

(ii) To construct a square equal in area to that of rectangle.

(iii) To measure sides of square and compare areas of rectangle and square.



Construction

(i) Take $m\overline{BC} = 4.5$ cm

(ii) At the end point B draw $\overrightarrow{BP} \perp \overline{BC}$
(iii) Cut off $m\overline{BA} = 2.2$cm
(iv) Complete the rectangle ABCD
(v) Produce $\overline{AD}$ to E making $m\overline{DE} = m\overline{DC}$.
(vi) Bisect $\overline{AE}$ at O.
(vii) With centre O and radius $m\overline{OA}$ describe a semi circle.
(viii) Produce $\overline{CD}$ to meet the semi-circle in M.
(ix) On $\overline{DM}$ as a side construct a square DFLM. This shall be the required square.
(x) Measure the side of the square $m\overline{DF} = 3.2$ cm. Area of square $= 3.2 \times 3.2 = 10.24$ cm $= 10$ cm (approx). Area of rectangle $= 4.5 \times 2.2 = 9.90$ cm^2 $= 10$ cm (approx.)

3. In Q.2 above verify by measurement that the perimeter of the square is less than that of the rectangle.

Ans: Perimeter of square $= 4 \times 3.2 = 12.8$ cm.
Perimeter of rectangle $= 2(4.5 + 2.2) = 2(6.7) = 13.4$cm
So perimeter of square is less than that of rectangle.

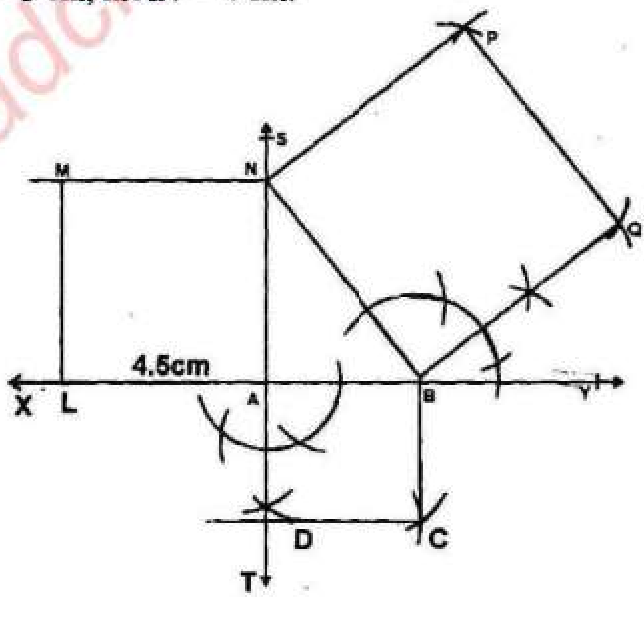
4. Construct a square equal in area to the sum of two squares having sides 3 cm and 4 cm respectively.

Given: Sides of two squares, $m\overline{AB} = 3$ cm, $m\overline{AN} = 4$ cm.

Required: To construct a square with area equal to the sum of the areas of the given two squares.

Construction

(i) Through a point A, draw two lines $\overleftrightarrow{XY}$ and $\overleftrightarrow{ST}$ perpendicular to each other.
(ii) Cut off $m\overline{AB} = 3$ cm and $m\overline{AL} = 4$cm
(iii) AB as a side of square complete the square ABCD.
(iv) $\overline{AL}$ as a side of square complete the square ALMN.
(v) Join $\overline{BN}$.
(vi) With $\overline{BN}$ as a side, complete the square BQPN.
BQPN is the required square:

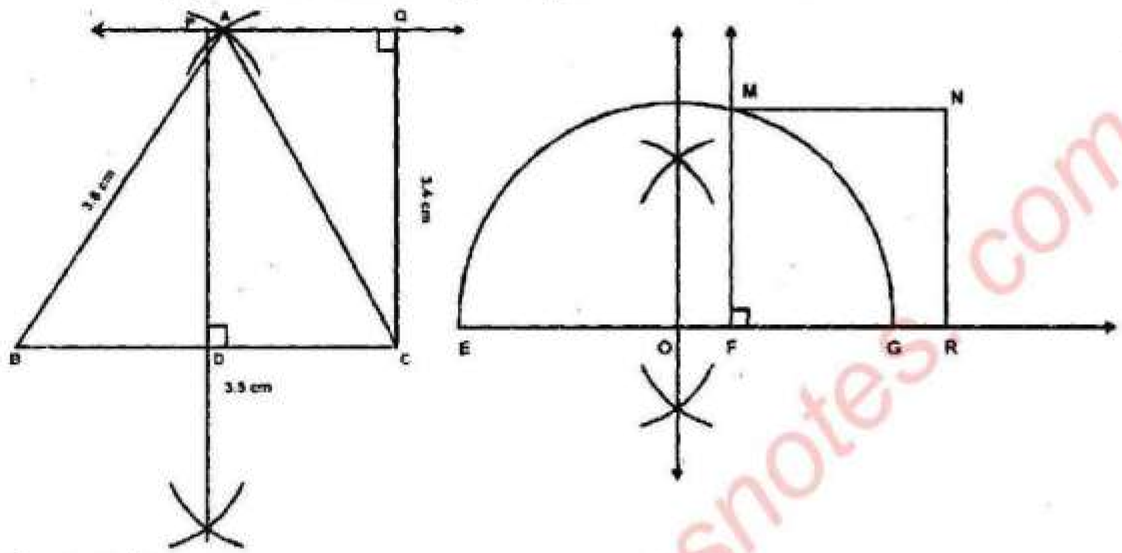


5. Construct a Δ having base 3.5 cm and other two sides equal to 3.4 cm and 3.8cm respectively. Transform it into a square of equal area.

Given: Sides $m\overline{BC} = 3.5$ cm, $m\overline{AB} = 3.8$ cm, $m\overline{AC} = 3.4$cm.

Required: (i) To construct ΔABC.

(ii) To construct a square equal in area to the ΔABC.



Construction

(i) Take $m\overline{BC} = 3.5$ cm.

(ii) With B as centre draw ail arc of radius 3.8 cm and with C as centre and radius 3.4 cm draw an other arc to meet the first arc at A.

(iii) Join A to B and A to C to complete the ΔABC.

(iv) Draw $\overleftrightarrow{PAQ} \parallel \overline{BC}$.

(v) Draw perpendicular bisector of $\overline{BC}$, bisecting it at D and meeting $\overleftrightarrow{PAQ}$ at P.

(vi) Draw $\overline{CQ} \perp \overline{PQ}$ meeting it in Q.

(vii) Take a line $\overline{EFG}$ and cut off $\overline{EF} = \overline{DP}$ and $\overline{FG} = \overline{DC}$.

(viii) Bisect $\overline{EG}$ at O.

(ix) With O as centre and radius $m\overline{OE}$ draw a semi-circle.

(x) At F draw $\overline{FM} \perp \overline{EG}$ meeting the semi circle at M.

(xi) With $\overline{MF}$ as a side complete the square FMNR.

6. Construct a Δ having base 5 cm and other sides equal to 5 cm and 6 cm. Construct a square equal in area to given Δ.

Given: $m\overline{BC} = 5$ cm, $m\overline{AB} = 6$ cm and $m\overline{AC} = 5$ cm.

Required: **(i)** to construct ΔABC.

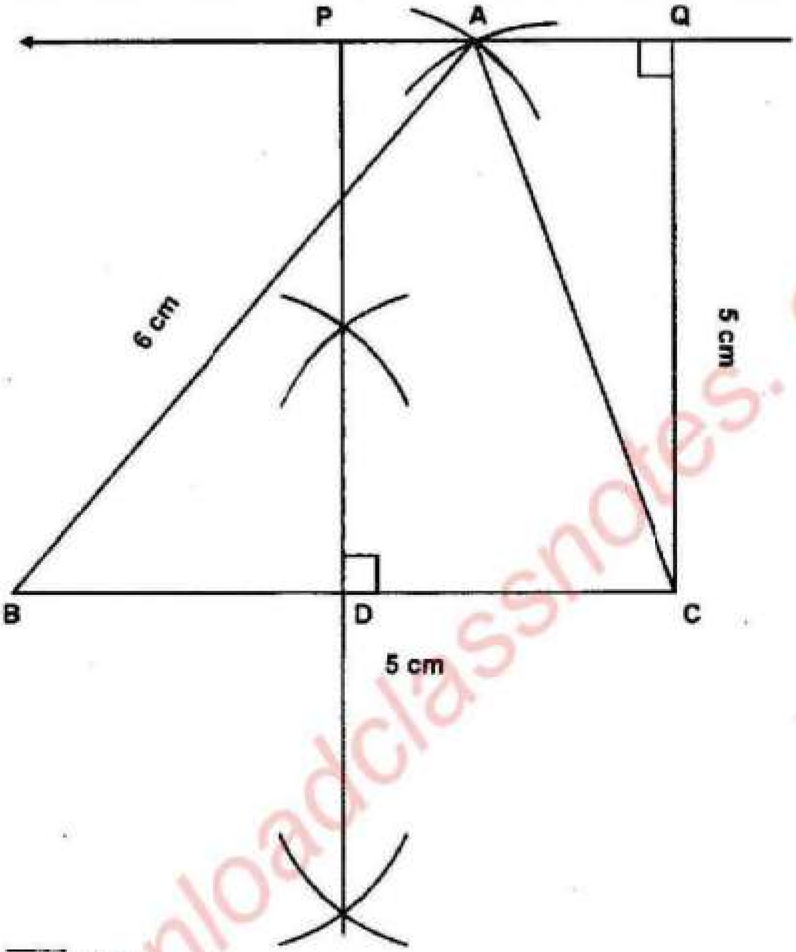
(ii) To construct a square equal in area to given A.

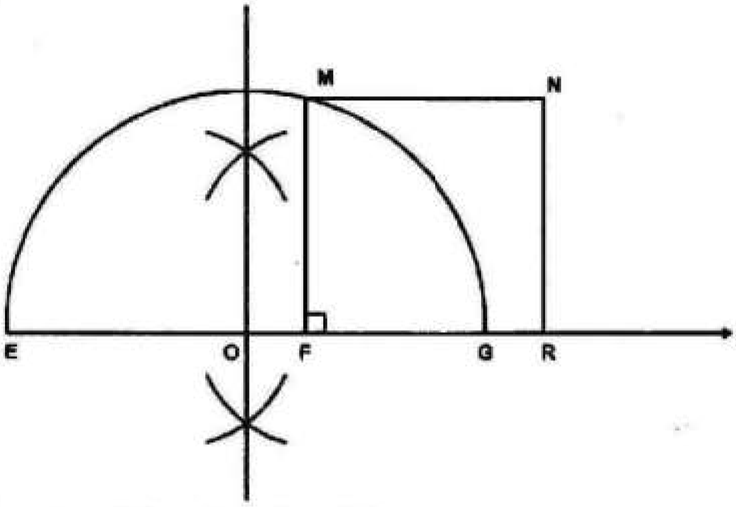
Construction

(i) Take $m\overline{BC} = 5$ cm.

(ii) With B as centre and radius. 6 cm, draw an arc, With centre at C and radius 5 cm draw an other arc to cut the first arc at A.

(iii) Join $\overline{AB}$ and $\overline{AC}$ to get the ΔABC.





(iv) Draw $\overline{PAQ} \parallel BC$.

(v) Draw perpendicular bisector of $\overline{BC}$, bisecting it at D and meeting $\overline{PAQ}$ at P.

(vi) Draw $\overline{CQ} \perp \overline{PQ}$ meeting it in Q.

(vii) Take a line $\overline{EFG}$ and cut off $\overline{EF} = \overline{DP}$ and $\overline{FG} = \overline{DC}$.

(viii) Bisect $\overline{EG}$ at O.

(ix) With O as centre and radius $m\overline{OE}$ draw a semi-circle.

(x) At F draw $\overline{FM} \perp \overline{EG}$ meeting the semi-circle at M.

(xi) With $\overline{MF}$ as a side complete required square FMNR.

Review Exercise 17

1. Fill in the following blanks to make the statement true:

(i) The side of a right angled triangle opposite to 90° is called......

(ii) The line segment joining a vertex of a triangle to the mid-point of its opposite side is called a ......

(iii) A line drawn from a vertex of a triangle which is ...... to its opposite side is called an altitude of the triangle.

(iv) The bisectors of the three angles of a triangle are.......

(v) The point of concurrency of the right bisectors of the three sides of the triangle is ...... from its vertices.

(vi) Two or more triangles are said to be similar if they are equiangular and measures of their corresponding sides are.......

(vii) The altitudes of a right triangle are concurrent at the ...... of the right angle.

Solution:

(i) hypotenuse (ii) median (iii) perpendicular
(iv) concurrent (v) equidistant (vi) proportional
(vii) vertex

2. Multiple Choice Questions. Choose the correct answer.

(i) A triangle having two sides congruent is called......

(a) scalene (b) right angled
(c) equilateral (d) isosceles

(ii) A quadrilateral having each angle equal to 90° is called ....

(a) parallelogram (b) rectangle
(c) trapezium (d) rhombus

(iii) The right bisectors of the three sides of a triangle are ......

(a) congruent (b) collinear
(c) concurrent (d) parallel

(iv) The ...... altitudes of an isosceles triangle are congruent.

(a) two (b) three
(c) four (d) none

(v) A point equidistant from the end points of a line-segment is on its ......

(a) bisector (b) right-bisector
(c) perpendicular (d) median

(vi) ...... congruent triangles can be made by joining the mid-points of the sides of a triangle.

(a) three (b) four
(c) five (d) two

(vii) The diagonals of a parallelogram ...... each other.
(a) bisect (b) trisect
(c) bisect at right angle (d) none of these

(viii) The medians of a triangle cut each other in the ratio......
(a) 4 : 1 (b) 3 : 1
(c) 2 :1 (d) 1 : 1

(ix) One angle on the base of an isosceles triangle is 30°. What is the measure of its vertical angle?
(a) 30° (b) 60°
(c) 90° (d) 120°

(x) If the three altitudes of a triangle are congruent, then the triangle is ......
(a) equilateral (b) right angled
(c) isosceles (d) acute angled

(xi) If two medians of a triangle are congruent then the triangle will be ......
(a) isosceles (b) equilateral
(c) right angled (d) acute angled

Solution:

(i) d **(ii)** b **(iii)** c **(iv)** a **(v)** b **(vi)** b
(vii) a **(viii)** c **(ix)** d **(x)** a **(xi)** a

3. Define the following
(i) Incentre **(ii) Circumcentre**
(iii) Ortho centre **(iv) Centroid**
(v) Point of concurrency

Solution: (i) Incentre

The internal bisectors of the angles of a triangle meet at a point called the incentre of the triangle.

(ii) Circumcentre

The point of concurrency of the three perpendicular bisectors of the sides of a Δ is called the circumcentre of the Δ.

(iii) Ortho-centre

The point of concurrency of the three altitudes of a Δ is called its orthocentre.

(iv) Centroid

The point where the three medians of a Δ meet is called the centroid of the triangle.

(v) Point of concurrency

Three or more than three lines are said to be concurrent, if they all pass through the same point. The common point is called the point of concurrency of the lines.